ARTICLE IN PRESS
JID: CHAOS [m5G;March 30, 2017;19:5]

Chaos, Solitons and Fractals 000 (2017) 1–6

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

A generalization of the Lomnitz logarithmic creep law via Hadamard fractional calculus

Roberto Garra[a], Francesco Mainardi[b,*], Giorgio Spada[c]

[a] Dipartimento di Scienze Statistiche, University of Rome "La Sapienza", Italy
[b] Dipartimento di Fisica e Astronomia (DIFA), University of Bologna "Alma Mater Studiorum", and INFN, Italy
[c] Dipartimento di Scienze Pure e Applicate (DiSPeA), University of Urbino "Carlo Bo", Italy

A R T I C L E I N F O

Article history:
Received 19 January 2017
Revised 13 March 2017
Accepted 14 March 2017
Available online xxx

MSC:
26A33
45D05
74D05
74L10
76A10

Keywords:
Linear viscoelasticity
Creep
Relaxation
Hadamard fractional derivative
Fractional calculus
Volterra integral equations
Ultra slow kinetics

A B S T R A C T

We present a new approach based on linear integro-differential operators with logarithmic kernel related to the Hadamard fractional calculus in order to generalize, by a parameter $\nu \in (0, 1]$, the logarithmic creep law known in rheology as Lomnitz law (obtained for $\nu = 1$). We derive the constitutive stress-strain relation of this generalized model in a form that couples memory effects and time-varying viscosity. Then, based on the hereditary theory of linear viscoelasticity, we also derive the corresponding relaxation function by solving numerically a Volterra integral equation of the second kind. So doing we provide a full characterization of the new model both in creep and in relaxation representation, where the slow varying functions of logarithmic type play a fundamental role as required in processes of ultra slow kinetics.

1. Introduction

In 1956 Lomnitz [15] introduced a logarithmic creep law to treat the creep behaviour of igneous rocks. The original Lomnitz creep law provides the strain response $\epsilon(t)$ to a constant stress $\sigma(t) = \sigma_0$ for $t \geq 0$ in the form

$$\epsilon(t) = \frac{\sigma_0}{E_0}[1 + q\ln(1 + t/\tau_0)], \quad t \geq 0\,, \tag{1.1}$$

where E_0 is the shear modulus, $\tau_0 > 0$ is a characteristic time during which the transition from elastic to creep-type deformation occurs and q is a positive non-dimensional constant. The logarithmic creep law suggested by Lomnitz on the basis of an empirical reasoning has found many applications in other papers of the same author [16,17].

In 1958 Jeffreys [11] proposed a power law of creep, generalizing the Lomnitz logarithmic law to broaden the geophysical applications to fluid-like materials including igneous rocks. This generalized law, however, can be applied also to solid-like viscoelastic materials as shown in a revisited version in 1984 by Strick [35]. More recently, in 2012 Mainardi and Spada [23] have provided a full characterization of the rheological properties related to this general model thus including the original Lomnitz creep law.

We recall that in the linear theory of viscoelasticity, based on the hereditary theory by Volterra, a viscoelastic body is characterized by two distinct but interrelated material functions, causal in time (i.e. vanishing for $t < 0$): the creep compliance $J(t)$ (the strain response to a unit step of stress) and the relaxation modulus $G(t)$ (the stress response to a unit step of strain). For more details see e.g. the treatises by Christensen [5], Pipkin [28] and Mainardi [19].

By taking $J(0^+) = J_0 > 0$ so that $G(0^+) = G_0 = 1/J_0$, the body is assumed to exhibit a non vanishing instantaneous response both in the creep and in the relaxation tests. As a consequence, we find it convenient to introduce two non-dimensional quantities $\psi(t)$ and $\phi(t)$ as follows

$$J(t) = J_0[1 + \psi(t)]\,, \quad G(t) = G_0\,\phi(t)\,, \tag{1.2}$$

* Corresponding author.
E-mail addresses: roberto.garra@sbai.uniroma1.it (R. Garra), francesco.mainardi@bo.infn.it (F. Mainardi), giorgio.spada@gmail.com (G. Spada).

http://dx.doi.org/10.1016/j.chaos.2017.03.032
0960-0779/

where $\psi(t)$ is a non-negative increasing function with $\psi(0) = 0$ and $\phi(t)$ is a non-negative decreasing function with $\phi(0) = 1$. Henceforth, $\psi(t)$ and $\phi(t)$ will be referred to as dimensionless creep function and relaxation function, respectively. Viscoelastic bodies can be distinguished in solid-like and fluid-like whether $J(+\infty)$ is finite or infinite so that $G(+\infty) = 1/J(+\infty)$ is non zero or zero, correspondingly.

It is quite common in linear viscoelasticity to require the existence of positive retardation and relaxation spectra for the material functions $J(t)$ and $G(t)$, as pointed out by Gross in his 1953 monograph on the mathematical structure of the theories of viscoelasticity [7]. This implies, as formerly proved in 1973 by Molinari [26] and revisited in 2005 by Hanyga [9], see also Mainardi's book [19], that $J(t)$ and $G(t)$ turn out to be Bernstein and Completely Monotonic functions, respectively. For their mathematical properties the interested reader is referred to the excellent monograph by Schilling et al. [34].

As pointed out e.g. in [19], the relaxation modulus $G(t)$ can be derived from the corresponding creep compliance $J(t)$ through the Volterra integral equation of the second kind

$$G(t) = \frac{1}{J_0} - \frac{1}{J_0}\int_0^t \frac{dJ}{dt'} G(t-t')\, dt'\,; \tag{1.3}$$

as a consequence, the dimensionless relaxation function $\phi(t)$ obeys the Volterra integral equation

$$\phi(t) = 1 - \int_0^t \frac{d\psi}{dt}\phi(t-t')dt'\,. \tag{1.4}$$

Mainardi and Spada in [23] have shown, both analytically and numerically, that the relaxation function corresponding to the Lomnitz creep law decays in time as the slow varying function $1/\ln t$.

Quite recently Pandey and Holm [27] have discussed the meaning of the empirical Lomnitz logarithmic law in the framework of time-dependent non-Newtonian rheology, where the stress-strain relation is

$$\sigma(t) = \eta(t)\dot{\epsilon}(t)\,, \quad t \ge 0\,, \tag{1.5}$$

where $\eta(t) > 0$ represents a time-dependent viscosity coefficient. In particular, they have shown that the stress-strain equation leading to the Lomnitz law is

$$\frac{q}{E_0}\,\sigma(t) = \left(1 + \frac{t}{\tau_0}\right)\dot{\epsilon}(t), \quad t \ge 0\,, \tag{1.6}$$

so that the time evolution of viscosity is represented by the differential operator

$$\widehat{O}_1^t := \left(1 + \frac{t}{\tau_0}\right)\frac{d}{dt}\,. \tag{1.7}$$

The starting point of Pandey and Holm [27] is a spring-dashpot viscoelastic model (of Maxwell type) with viscosity varying linearly in time that provides a relaxation function decaying in time to zero as a negative power law. We note that the stress-strain relation (1.6) indeed yields the Lomnitz creep law (1.1) for constant stress $\sigma(t) = \sigma_0$, but the power law for the relaxation function is not compatible with its ultra-slow decay of logarithmic type derived by Mainardi and Spada in [23] in the framework of the linear theory of viscoelasticity as solution of the Volterra integral Eq. (1.3).

In Geophysics there exists another approach to derive the Lomnitz creep law: it is due to Scheidegger [32,33], who in 1970 proposed a non-linear stress-strain relation that reads

$$\dot{\sigma}(t) = 2\eta\,\ddot{\epsilon}(t) + \beta\,\dot{\epsilon}^2(t)\,, \tag{1.8}$$

where η is the (constant) viscosity and β a creep factor. The integration of this differential equation for a step input of stress $\sigma(t) = \sigma_0$ for $t \ge 0$ leads to the Lomnitz creep law (1.1) provided that $2\eta/\beta = q\sigma_0/E_0$ and τ_0 is a suitable time constant of integration. This non-linear approach, however, even if justified by the author for some effects related to energy dissipation in rocks, has not found a validation in the literature up to nowadays. Furthermore, no investigation for the relaxation of stress under constant strain has been considered.

The above discussion implies that in order to justify the Lomnitz creep law we have (at least, to our knowledge) three possible ways: the standard one based on the linear hereditary Volterra theory, the approach with non constant viscosity followed by Pandey and Holm [27] and the non-linear approach proposed by Scheidegger.

In this paper, starting with the Pandey and Holm [27] approach, we set up an iterative operational method based on operator (1.7) which leads to the generalized Lomnitz law

$$\epsilon(t) = \frac{\sigma_0}{E_0}\left[1 + q\frac{\ln^\nu\left(1+\frac{t}{\tau_0}\right)}{\Gamma(1+\nu)}\right], \quad 0 < \nu \le 1\,, \quad t \ge 0\,. \tag{1.9}$$

We will then derive the corresponding relaxation function of this law by solving the related Volterra integral equation. An interesting outcome of our analysis is that the resulting rheological model considers both a time varying viscosity and the memory effects required by the hereditary theory of linear viscoelasticity.

We first give some mathematical preliminaries in the next Section and then in Section 3 we explain the meaning of our approach. In Section 4 we discuss the implications of our work and finally conclusions are drawn in Section 5.

For readers' convenience we add two appendices. In Appendix A we recall the essentials of the Hadamard fractional calculus on which our operational approach is based. In Appendix B we outline the numerical method adopted to solve the Volterra integral equation satisfied by the relaxation function of our generalized model.

2. Integro-differential operators with logarithmic kernels

In the recent paper by Beghin, Garra and Macci [2], an integro-differential operator with logarithmic kernel has been introduced in the context of correlated fractional negative binomial processes in statistics. Using their notation, the time-evolution operator $\widehat{O}_\nu^t$ acting on a sufficiently well-behaved function $f(t)$ is defined as

$$\widehat{O}_\nu^t f(t) := \frac{1}{\Gamma(n-\nu)} \times \int_{\frac{1-a}{b}}^t \ln^{n-1-\nu}\left(\frac{a+bt}{a+b\tau}\right) \times \left[\left(\left(\frac{a}{b}+\tau\right)\frac{d}{d\tau}\right)^n f(\tau)\right]\frac{b}{a+b\tau}\,d\tau, \tag{2.1}$$

for $n-1 < \nu < n \in \mathbb{N}$, $0 < a \le 1$ and $b > 0$.

A relevant property of this operator is given by the following result

$$\widehat{O}_\nu^t \ln^\beta(a+bt) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)}\ln^{\beta-\nu}(a+bt) \tag{2.2}$$

for $\nu \in (0,1)$ and $\beta > -1 \setminus \{0\}$, see [2], pag. 1057 for the details. Moreover we have that $\widehat{O}_\nu^t\ const. = 0$. We refer to the Appendix A for a short survey about fractional-type operators with logarithmic kernel, starting from the so-called Hadamard fractional calculus.

In analogy with the classical theory of fractional calculus (see e.g. the monograph by Kilbas, Srivastava and Trujillo [13]), we introduce the integral operator with logarithmic kernel acting on a sufficiently well-behaved function $f(t)$ as

$$\widehat{I}_\alpha^t f(t) := \frac{1}{\Gamma(\alpha)}\int_{\frac{1-a}{b}}^t \ln^{\alpha-1}\left(\frac{a+bt}{a+b\tau}\right) f(\tau)\frac{b}{a+b\tau}\,d\tau,\ \alpha > 0, \tag{2.3}$$