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A search for a spectral technique to solve nonlinear fractional differential equations

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ABSTRACT

A spectral decomposition method is used to obtain solutions to a class of nonlinear differential equations. We extend this approach to the analysis of the fractional form of these equations and demonstrate the method by applying it to the fractional Riccati equation, the fractional logistic equation and a fractional cubic equation. The solutions reduce to those of the ordinary nonlinear differential equations, when the order of the fractional derivative is $\alpha = 1$. The exact analytic solutions to the fractional nonlinear differential equations had not been previously known, so we evaluate how well the derived solutions satisfy the corresponding fractional dynamic equations. In the three cases we find a small, apparently generic, systematic error that we are not able to fully interpret.

1. Introduction

Herein we propose a spectral method for solving fractional nonlinear rate equations of a certain kind. The method is not perturbative, but neither is it exact, since it gives rise to systematic deviations of the analytic solution from the numerical solution at intermediate times that reaches a maximum value of 2%. On the one hand, the spectral method provides a remarkable good approximation to the solution obtained through numerical integration. On the other hand, the source of the small but systematic deviation from the numerical solution remains a mystery. This paper presents the approach in detail and introduces a new problem that requires explanation.

Despite the advances made into the understanding of complex nonlinear systems in the last half of the twentieth century, many physical phenomena failed to be described using the tools of ordinary calculus. Nonlocal distributed effects and memory effects observed in relaxation phenomena [1], living systems [2,3], wave propagation in porous materials [4] have been more successfully modeled within the framework of the fractional calculus [5]. Fractional differential equations (FDE) have been adopted to explain these and other complex phenomena [6,7]. Since exact solutions to the majority of FDEs are not available, the search for appropriate analytical and numerical methods is a subject of ongoing research. Recently, a number of approaches devoted to solving FDEs have been proposed. Examples include Adomian decomposition method [8], homotopy perturbation method [9,10], the fractional subequation method. and the Haar wavelet method [11], to name but a few. However, the convergence region of solutions obtained with these algorithms is rather small.

It was hypothesized in [12] that the spectral decomposition method can be extended to the analysis of a class of nonlinear fractional differential equations (NFDEs). Herein we demonstrate the method by applying it to the fractional Riccati equation (FRE), the fractional logisticequation (FLE) and a fractional cubic equation (FCE), where the fractional-order is in the range $0 < \alpha \leq 1$. The solutions obtained are shown to have the correct short-time and long-time behaviors. Additionally, they reduce to the well known solutions of the ordinary nonlinear differential equations, when the order of the fractional derivative is $\alpha = 1$. In the cases considered herein the exact analytic solution to the FNDE was not known previously, we evaluate how well the derived solutions satisfy the corresponding NFDE using numerical techniques. In all cases we find a very small, but systematic deviation of the analytic from the numerical solutions that has eluded our best efforts to interpret. One possibility, of course, is that the numerical technique used to solve the NFDE is the culprit, since it was based on numerically solving linear fractional equations. But this remains to be investigated.

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In Section 2 we introduce the spectral decomposition of the solution to define the eigenvalue problem for integer-order linear and nonlinear rate equations, as well as NFDEs. In Section 3 we obtain a series expansion over the spectrum of eigenvalues and eigenfunctions for the solution to three NFDEs, where the exponentials in the solutions to the integer-order equations, also obtained, are replaced with Mittag-Leffler functions (MLFs). Exact solutions to NFDEs are rare in the literature [13,14], so to test the validity of the analytic results we numerically evaluate how the solutions obtained satisfy the appropriate NFDE. To our surprise the error function measuring this fit is not zero, but varies in time, increasing as $t^{2\alpha}$ at early times and decreasing as $t^{-\alpha}$ at late times, and reaching a maximum difference of less than a few percent at an intermediate time. This non-monotonic scaling difference is shown to occur with the solutions to the FRE, the FLE, as well as, the FCE, all with the same qualitative behavior in the error. In Section 8 we draw some tentative conclusions including the speculation that this systematic deviation may be generic.

2. Spectral decomposition

2.1. Integer operator

Let us begin by establishing the nomenclature used in the study of the nonlinear differential equations. Consider the one-dimensional first-order differential equation

$$\frac{d}{dt}X(t) = \mathcal{O}X(t), \tag{1}$$

where $X(t)$ is the dynamic variable of interest and $\mathcal{O}$ is a generic operator acting on $X(t)$. Allowing Eq. (1) to describe any dynamical system of interest entails the formal solution

$$X(t) = e^{\mathcal{O}_0 t}x_0, \tag{2}$$

where $x_0 \equiv X(0)$ defines the initial condition in the phase space for the dynamic variable and the operator $\mathcal{O}_0$ acts on the initial condition. The exponential operator is formally defined by the series expansion

$$e^{\mathcal{O}_0 t} = \sum_{k=0}^{\infty}\frac{(\mathcal{O}_0 t)^k}{\Gamma(k+1)} \tag{3}$$

so that the solution Eq. (2) can be expressed as

$$X(t) = \sum_{k=0}^{\infty}\frac{(\mathcal{O}_0 t)^k}{\Gamma(k+1)}x_0 = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k+1)}{\mathcal{O}_0}^k x_0 \tag{4}$$

where the operator ${\mathcal{O}_0}^k$ acts solely on the initial condition. Note that for a linear equation with a constant rate λ the operator is given by

$${\mathcal{O}_0}^k x_0 = \left(\lambda x_0 \frac{\partial}{\partial x_0}\right)^k x_0 = \lambda^k x_0, \tag{5}$$

which when inserted into Eq. (4) and summing the series yields the exponential solution to the scalar rate equation

$$X(t) = e^{\lambda t}x_0. \tag{6}$$

It is apparent that Eq. (5) has a form suggestive of an eigenvalue equation and that the solution to the general integer-operator rate equation can be expressed as an eigenfunction expansion over the spectrum of eigenvalues

$$X(t) = \sum_{k=0}^{\infty}C_k\phi_k(x_0)\chi_k(t). \tag{7}$$

The quantity $\phi_k(x_0)\chi_k(t)$ is the eigenfunction, factored into a piece determined by the spectrum of eigenvalues $\{\lambda_k; k = 0, 1, 2, \cdot\cdot\}$, a piece determined by the initial condition x_0, and the expansion coefficient C_k determined by the dynamics and overall initial normalization. Inserting Eq. (7) into (1), allows us to separate out the time-dependence of the components of the expansion

$$\frac{d}{dt}\chi_k(t) = \lambda_k\chi_k(t) \Rightarrow \chi_k(t) = e^{\lambda_k t}. \tag{8}$$

Correspondingly, the eigenvalue equations are given by

$$\mathcal{O}_0\phi_k(x_0) = \lambda_k\phi_k(x_0) \tag{9}$$

and the eigenvalues are determined by the form of the operator.

In the linear case just considered the operator is the same as before, so the equation for the eigenfunction is

$$\lambda x_0\frac{d\phi_k}{dx_0} = \lambda_k\phi_k$$

with the solution

$$\phi_k(x_0) = x_0^{\frac{\lambda_k}{\lambda}}.$$

The linear eigenvalue spectrum is degenerate $\lambda_k = \lambda$ and the coefficients are determined from the initial condition to satisfy

$$\sum_{k=0}^{\infty}C_k = 1.$$

The resulting solution is, of course, given by Eq. (6).

2.2. Non-integer (fractional) operator

Now we assume that this general form of a solution to a differential equation translates to the fractional calculus domain. Thus, we replace Eq. (1) with the fractional differential equation

$$\frac{d^\alpha}{dt^\alpha}X(t) = \mathcal{O}X(t), \tag{10}$$

where $0 < \alpha \leq 1$. We assume the fractional derivative to be defined in the Caputo sense:

$$\frac{d^\alpha}{dt^\alpha}X(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{X'(\tau)}{(t-\tau)^\alpha}d\tau. \tag{11}$$

where $X'(\tau)$ denotes the derivative of $X(\tau)$ with respect to its argument. Eq. (10) can be solved analytically in terms of the MLF by employing the spectral decomposition introduced above in which case we have for the components of the eigenfunction

$$\frac{d^\alpha}{dt^\alpha}\chi_k(t) = \lambda_k\chi_k(t), \tag{12}$$

to obtain the MLF evaluated over the spectrum of eigenvalues

$$\chi_k(t) = E_\alpha(\lambda_k t^\alpha). \tag{13}$$

The MLF is defined by the series [15,16]

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}. \tag{14}$$

Consequently, inserting the MLF into the expansion for the solution yields [12]

$$X(t) = \sum_{k=0}^{\infty}C_k\phi_k(x_0)E_\alpha(\lambda_k t^\alpha) \tag{15}$$

and the eigenvalues are determined by the operator in Eq. (9). The MLF reduces to an exponential function when $\alpha = 1$, reducing the series expansion to the ordinary solution of Eq. (2) in that case.

Note that we can adopt the same formal spectral decomposition employed for the integer-derivative case discussed above for the fractional-order dynamics considered here. But, before we explore the fractional case, let us examine an integer-order nonlinear dynamic equation.