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Multifractal analysis of weighted local entropies

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In this paper, we give the multifractal analysis of the weighted local entropies for arbitrary invariant measures. Our result is applied to self-affine systems.

1. Background and introduction

Let (X, d, T) be a dynamical system, where (X, d) is a compact metric space and $T: X \to X$ is a continuous map. The set $M(X)$ of all Borel probability measures is compact under the weak* topology. Denote by $M(X, T) \subset M(X)$ the subset of all T-invariant measures and $E(X, T) \subset M(X, T)$ the subset of all ergodic measures. Multifractal analysis is concerned with the study of pointwise dimension of a Borel measure μ (provided the limit exists):

$$d_\mu(x) = \lim_{\epsilon \to 0} \frac{\log \mu(B(x, \epsilon))}{\log \epsilon},$$

where $B(x, \epsilon)$ is an open ϵ-neighborhood of x. Set

$$X_\alpha := \{x \in X : d_\mu(x) = \alpha\}.$$

The purpose is to describe the set X_α. It is worthwhile to mention that the multifractal analysis of Birkhoff average is closely related to the pointwise dimension of the Borel measure. We refer the reader to Refs. [4,10,12,14,16]. Here, we can introduce the general form of Pesin's multifractal formalism in [8], or [2] as follows. Consider a function $g : Y \to [-\infty, +\infty]$ in a subset Y of X. The level set

$$K_\alpha^g = \{x \in Y : g(x) = \alpha\}$$

are pairwise disjoint, and we obtain a *multifractal decomposition* of X given by

$$X = (X \setminus Y) \cup \bigcup_{\alpha \in [-\infty, +\infty]} K_\alpha^g.$$

Let G be a function defined in the set of subsets of X. The *multifractal spectrum* : $\mathcal{F} : [-\infty, +\infty] \to \mathbb{R}$ of the pair (g, G) is defined by

$$\mathcal{F}(\alpha) = G(K_\alpha^g),$$

where g may denote the Birkhoff averages, Lyapunov exponents, pointwise dimension or local entropies and G may denote the topological entropy, topological pressure or Hausdorff dimension. For fixed $q \in \mathbb{R}$ and $\mu \in M(X)$, Olsen [7] defined a generalized Hausdorff dimension $\dim_\mu^q(\cdot)$ for $q \in \mathbb{R}$ (for the detailed definitions, see Section 3) and established the relation formula of dimensions. And then in [6], Olsen studied self-affine multifractal analysis in $\mathbb{R}^d$ by using the formalism introduced in [7] with separation condition. We state these results as follows:

- Let μ be a cookie-cutter measure in $\mathbb{R}$ or graph directed measure in $\mathbb{R}^d$ with totally disconnected support. Then
$$\dim_H(X_\alpha) = \inf_q\{q\alpha + \dim_\mu^q(\operatorname{supp}\mu)\}.$$
- Let μ denote the self-affine Sierpinski Sponge measure. Then
$$\dim_H(X_\alpha) = \inf_q\{q\alpha + \dim_\mu^q(\operatorname{supp}\mu)\}.$$

In dynamical systems, the dynamical ball is always studied instead of the geometry ball. More precisely, for $x \in X$, we define the dynamical ball $B_n(x, \epsilon)$ by

$$B_n(x, \epsilon) := \{y \in X : d(T^j x, T^j y) < \epsilon, 0 \le j \le n - 1\}.$$

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We define the *low(resp.upper)local(pointwise)entropies* as follows:

$$\underline{h}_\mu(T,x)=\lim_{\epsilon\to 0}\liminf_{n\to\infty}-\frac{1}{n}\log\mu(B_n(x,\epsilon)),$$

$$\overline{h}_\mu(T,x)=\lim_{\epsilon\to 0}\limsup_{n\to\infty}-\frac{1}{n}\log\mu(B_n(x,\epsilon)).$$

Note that the limits exist as ϵ tends 0. We say that the local entropy exists at x if

$$\underline{h}_\mu(T,x)=\overline{h}_\mu(T,x).$$

In this case the common value will be denoted by $h_\mu(T, x)$. And then, for $\mu \in M(X, T)$ and $\alpha \geq 0$, define

$$\widehat{K}_\alpha(\mu)=\{x\in X : h_\mu(T,x)=\alpha\}.$$

In [13], Takens and Verbitski defined the (q, μ)-entropy $h_\mu(T, q, \cdot)$ by extending the definition of generalized Hausdorff dimension $\dim_\mu^q(\cdot)$ and showed the following formula:

$$h_{top}(\widehat{K}_\alpha(\mu))=q\alpha+h_\mu(T,q,\widehat{K}_\alpha(\mu)),$$

where $h_{top}(\cdot)$ denotes the topological entropy. Later, in 2007, Yan and Chen [15] considered the multifractal spectra associated with Poincaré recurrences and established an exact formula on multifractal spectrum of local entropies for recurrence time.

A natural question is that how does this work without the separation condition? In this paper, we will study the self-affine multifractal in general topological dynamical systems using the weighted entropy introduced in [5] without the separation condition in [6].

2. Preliminaries and main results

In [5], Feng and Huang introduced the weighted entropy for factor maps between general topological dynamical systems. Let $k \geq 2$. Assume that $(X_i, d_i), i = 1, \ldots, k$, are compact metric spaces, and (X_i, T_i) are topological dynamical systems. Moreover, assume that for each $1 \leq i \leq k-1$, (X_{i+1}, T_{i+1}) is a factor of (X_i, T_i) with a factor map $\pi_i : X_i \to X_{i+1}$; in other words $\pi_1, \ldots, \pi_{k-1}$ are continuous maps such that the following diagrams commute:

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{\pi_1} & X_2 & \xrightarrow{\pi_2} & \cdots & \xrightarrow{\pi_{k-1}} & X_k\\
T_1\downarrow & & T_2\downarrow & & & & \downarrow T_k\\
X_1 & \xrightarrow{\pi_1} & X_2 & \xrightarrow{\pi_2} & \cdots & \xrightarrow{\pi_{k-1}} & X_k
\end{array}$$

For convenience, we use π_0 be the identity map on X_1. Define $\tau_i : X_1 \to X_{i+1}$ by $\tau_i = \pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_0$, for $i = 0, 1, \ldots, k-1$. Let $M(X_i, T_i)$ denote the set of all T_i-invariant Borel probability measures on X_i and $E(X_i, T_i)$ denote the set of ergodic measures. Fix $\mathbf{a} = (\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k}) \in \mathbb{R}^{\mathbf{k}}$ with $a_1 > 0$ and $a_i \geq 0$ for $i \geq 2$. For $\mu \in M(X_1, T_1)$, we call

$$h_\mu^{\mathbf{a}}(T_1):=\sum_{i=1}^{k}a_i h_{\mu\circ\tau_{i-1}^{-1}}(T_i)$$

the a-weighted measure-theoretic entropy of μ with respect to T_1, or simply, the a-weighted entropy of μ, where $h_{\mu\circ\tau_{i-1}^{-1}}(T_i)$ denotes the measure-theoretic entropy of $\mu\circ\tau_{i-1}^{-1}$ with respect to T_i.

Definition 2.1 (a-weighted Bowen ball)**.** For $x \in X_1, n \in \mathbb{N}, \epsilon > 0$, let

$$B_n^{\mathbf{a}}(x,\epsilon):=\{y\in X_1 : d_i(T_i^j\tau_{i-1}x, T_i^j\tau_{i-1}y) < \epsilon \text{ for } 0\leq j\leq \lceil(a_1+\cdots+a_i)n\rceil-1, i=1,\ldots,k\}$$

where $\lceil u\rceil$ denotes the least integer $\geq u$. We call $B_n^{\mathbf{a}}(x,\epsilon)$ the nth **a**-weighted Bowen ball of radius ϵ centered at x.

Remark 2.1. Return back to the metric spaces (X_i, d_i) and topological dynamical systems $(X_i, T_i), i = 1, 2, \ldots, k$. For $n \in \mathbb{N}$, define a metric $d_n^{\mathbf{a}}$ on X_1 by

$$d_n^{\mathbf{a}}(x,y)=\sup\{d_i(T_i^j\tau_{i-1}x, T_i^j\tau_{i-1}y) < \epsilon \text{ for } 0\leq j\leq \lceil(a_1+\cdots+a_i)n\rceil-1, i=1,\ldots,k\}.$$

Definition 2.2 [5]**.** Let $\mathbf{a}= (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$ with $a_1 > 0$, and $a_i \geq 0$ for $2 \leq i \leq k$. For any $n \in \mathbb{N}$, and $\epsilon > 0$, define

$$\mathcal{T}_{n,\epsilon}^{\mathbf{a}} := \{A\subset X_1 : A \text{ is a Borel subset of } B_n^{\mathbf{a}}(x,\epsilon) \text{ for some } x\in X_1\}.$$

For any subset $Z \subset X_1$, $s \geq 0$ and $N \in \mathbb{N}$, define

$$\Lambda^{\mathbf{a}}(Z,\epsilon,s,N)=\inf\sum_j \exp(-sn_j)$$

where the infimum is taken over all countable collections $\Gamma = \{(n_j, A_j)\}, n_j \geq N, A_j \in \mathcal{T}_{n_j,\epsilon}^{\mathbf{a}}$, and $\bigcup_{(n_j,A_j)\in\Gamma} A_j \supset Z$. The quantity $\Lambda^{\mathbf{a}}(Z, \epsilon, s, N)$ does not decrease with N, hence the following limit exists:

$$\Lambda^{\mathbf{a}}(Z,\epsilon,s)=\lim_{N\to\infty}\Lambda^{\mathbf{a}}(Z,\epsilon,s,N).$$

There exists a critical value of the parameters, which we will denote by $h^{\mathbf{a}}(Z, \epsilon)$, where $\Lambda^{\mathbf{a}}(Z, \epsilon, s)$ jumps from ∞ *to* 0, i.e

$$\Lambda^{\mathbf{a}}(Z,\epsilon,s)=\begin{cases}0 & if\ s>h^{\mathbf{a}}(Z,\epsilon)\\ \infty & if\ s<h^{\mathbf{a}}(Z,\epsilon).\end{cases}$$

Clearly, $h^{\mathbf{a}}(Z, \epsilon)$ does not decrease with ϵ, and hence the following limit exists:

$$h^{\mathbf{a}}(Z)=\lim_{\epsilon\to 0}h^{\mathbf{a}}(Z,\epsilon).$$

Definition 2.3. We define the weighted lower (upper) local (pointwise) entropies as follows:

$$\underline{h}_\mu^{\mathbf{a}}(T_1,x)=\lim_{\epsilon\to 0}\liminf_{n\to\infty}-\frac{1}{n}\log\mu(B_n^{\mathbf{a}}(x,\epsilon)),$$
$$\overline{h}_\mu^{\mathbf{a}}(T_1,x)=\lim_{\epsilon\to 0}\limsup_{n\to\infty}-\frac{1}{n}\log\mu(B_n^{\mathbf{a}}(x,\epsilon)).$$

We say that the weighted local entropy exists at x if

$$\underline{h}_\mu^{\mathbf{a}}(T_1,x)=\overline{h}_\mu^{\mathbf{a}}(T_1,x).$$

In this case the common value will be denoted by $h_\mu^{\mathbf{a}}(T_1,x)$. Similar to the Brin–Katok formula in [3], Feng and Huang [5] showed the weighted version of Brin–Katok formula as follows.

Theorem 2.1 [5]**.** *For each ergodic measure* $\mu \in M(X_1, T_1)$, *we have*

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{-\log\mu(B_n^{\mathbf{a}}(x,\epsilon))}{n}=\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{-\log\mu(B_n^{\mathbf{a}}(x,\epsilon))}{n}=h_\mu^{\mathbf{a}}(T_1)$$

for μ*-a.e.,* $x \in X_1$.

Let $\mu \in M(X_1, T_1)$ be an invariant Borel measure. For $\alpha \geq 0$, define

$$K_\alpha(\mu)=\{x\in X_1 : h_\mu^{\mathbf{a}}(T_1,x)=\alpha\}.$$

In this paper, we are interested in local entropies and spectra associated with the weighted Bowen ball. More precisely, we study the size of the set $K_\alpha(\mu)$.