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Exact solutions of space-time fractional EW and modified EW equations

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ABSTRACT

The bright soliton solutions and singular solutions are constructed for the space-time fractional EW and the space-time fractional modified EW (MEW) equations. Both equations are reduced to ordinary differential equations by the use of fractional complex transform (FCT) and properties of modified Riemann–Liouville derivative. Then, various ansatz method are implemented to construct the solutions for both equations.

1. Introduction

Several decades ago, more generalized forms of differential equations were described as fractional differential equations. Various phenomena in many natural and social science fields like engineering, geology, economics, meteorology, chemistry and physics are modeled by those equations [1,2]. The descriptions of diffusion, diffusive convection, Fokker–Plank type, evolution, and other differential equations are expanded by using fractional derivatives. Some well known fractional PDEs (FPDE) in literature can be listed as diffusion equation, nonlinear Schrödinger equation, Ginzburg–Landau equation, Landau–Lifshitz, Boussinesq equations, etc [2].

Even though there exist general methods for solutions of linear PDEs, the class of nonlinear PDEs has usually exact solutions. Sometimes it is also possible to obtain soliton-type solitary wave solution, which behaves like particles, that is, maintains its shape with constant speed and preserves its shape after collision with another soliton. The famous nonlinear PDEs having soliton solutions in literature are Korteweg-de Vries and Schrödinger equations. Soliton type solutions have great importance in optics, fluid dynamics, propagation of surface waves, and many other fields of physics and various engineering branches.

The integer ordered equation of the form

$$U_t(x,t) - U(x,t)U_x(x,t) - U_{xxt}(x,t) = 0 \tag{1}$$

was named as the Equal-width Equation (EWE) by Morrison et al. [3] due to having traveling wave solutions containing sech^2 function. The EWE has only lowest three polynomial conservation laws and they were determined in the same study. The single traveling wave solutions to the generalized form of the EWE are classified by implementing the complete discrimination system for polynomial [4]. Owing to having analytical solutions, the EWE also attracts many researchers studying numerical techniques for partial differential equations. So far, various numerical methods covering differential quadrature, Galerkin and meshless methods [5], lumped Galerkin method based on B-splines [6,7], septic B-spline collocation [8], spectral method [9], exponential cubic B-spline [10], moving least squares collocation [11], the method of lines based on meshless kernel [12] have been implemented to solve different problems constructed with the EWE.

Recently, parallel to developments in symbolic computations, lots of new techniques have been proposed to solve nonlinear PDEs exactly. Some of those methods covering the first integral, the sub-equation, Kudryashov, sine-cosine and ansatz methods have been applied for exact solutions for not only integer ordered and but also fractional ordered PDEs [13–20]. Some recent studies including various methods to solve FPDEs exactly in literature can be

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found in [21–38]. Besides these developments in the exact or analytical solution fields, some semi numerical techniques have also appeared to solve various fractional PDEs [39–43].

This study aims to generate exact solutions to the fractional equal-widthequation (FEWE) and the fractional modified fractional equal-width equation (FMEWE) of the forms

$$D_t^\beta u(x,t) + \epsilon D_x^\beta u^2(x,t) - \delta D_{xxt}^{3\beta} u(x,t) = 0 \tag{2}$$

and

$$D_t^\beta u(x,t) + \epsilon D_x^\beta u^3(x,t) - \delta D_{xxt}^{3\beta} u(x,t) = 0 \tag{3}$$

where ϵ and δ are real parameters and the modified Riemann–Liouville derivative (MRLD) operator of order β for the continuous function $u : \mathbb{R} \to \mathbb{R}$ defined as

$$D_x^\gamma u(x) = \begin{cases} \dfrac{1}{\Gamma(-\gamma)} \int\limits_0^x (x-\eta)^{-\gamma-1} u(\eta) d\eta, & \gamma < 0 \\ \dfrac{1}{\Gamma(-\gamma)} \dfrac{d}{dx} \int\limits_0^x (x-\eta)^{-\gamma} [u(\eta) - u(0)] d\eta, & 0 < \gamma < 1 \\ (u^{(p)}(x))^{(\gamma-p)}, \quad p \le \gamma \le p+1, & p \ge 1 \end{cases} \tag{4}$$

where the Gamma function is given as

$$\Gamma(\gamma) = \lim_{p\to\infty} \frac{p! p^\gamma}{\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+p)} \tag{5}$$

[44].

2. The properties of the MRLD and methodology of solution

Some properties of the MRLD can be listed as

$$D_x^\beta x^c = \frac{\Gamma(1+c)}{\Gamma(1+c-\beta)} x^{c-\beta}$$
$$D_x^\beta \{a w(x) + b v(x)\} = a D_x^\beta \{w(x)\} + b D_x^\beta \{v(x)\} \tag{6}$$

where a, b are constants and $c \in \mathbb{R}$ [45].

Consider the nonlinear FPDEs of the general implicit form

$$F(u, D_t^\beta u, D_x^\theta u, D_t^\beta D_t^\beta u, D_t^\beta D_t^\beta u, D_t^\beta D_x^\theta u, D_x^\theta D_x^\theta u, \ldots) = 0, \quad 0 < \beta, \theta < 1 \tag{7}$$

where β and θ are orders of the MRLD of the function $u = u(x,t)$. The FCT

$$u(x,t) = U(\zeta), \quad \zeta = \frac{\bar{k} x^\theta}{\Gamma(1+\theta)} - \frac{\bar{c} t^\beta}{\Gamma(1+\beta)} \tag{8}$$

where $\bar{k}$ and $\bar{c}$ are nonzero constants reduces (7) to an integer ordered ODE [46]. One should note that the chain rule can be calculated as

$$D_t^\beta u = \sigma_1 \frac{dU}{d\zeta} D_t^\beta \zeta$$
$$D_x^\beta u = \sigma_2 \frac{dU}{d\zeta} D_x^\beta \zeta \tag{9}$$

where σ_1 and σ_2 fractional indexes [47]. Substitution of the (8) into (7) and usage of chain rule defined (9) converts (7) to an ODE of the form

$$G\left(U, \frac{dU}{d\zeta}, \frac{d^2U}{d\zeta^2}, \ldots\right) = 0 \tag{10}$$

3. Solutions for fractional EW equation

Consider the FEWE equation

$$D_t^\beta u(x,t) + \epsilon D_x^\beta u^2(x,t) - \delta D_{xxt}^{3\beta} u(x,t) = 0 \quad t > 0, \quad 0 < \beta \le 1 \tag{11}$$

The use of the transformation (8) reduces the FEWE (11) to

$$-cU' + \epsilon k (U^2)' + \delta c k^2 U''' = 0 \tag{12}$$

where $k = \bar{k}\sigma_2$ and $c = \bar{c}\sigma_1$

3.1. Bright soliton solution

Let A, $\bar{k}$ and $\bar{c}$ be arbitrary constants. Then, assume that

$$U(\zeta) = A \operatorname{sech}^p \zeta, \quad \zeta = \frac{\bar{k} x^\beta}{\Gamma(1+\beta)} - \frac{\bar{c} t^\beta}{\Gamma(1+\beta)} \tag{13}$$

solves Eq. (12). Substituting this solution into the Eq. (12) leads to

$$(-\delta c p k^2 A - \delta c p^2 k^2 A) \operatorname{sech}^{p+2} \zeta + \epsilon k A^2 \operatorname{sech}^{2p} \zeta + (\delta c k^2 p^2 - cA) \operatorname{sech}^p \zeta = 0 \tag{14}$$

Assuming the powers are equal to each other such as $p + 2 = 2p$, p is determined as 2. Substituting this p value into Eq. (14) reduces it to

$$\left(-6\delta c k^2 A + \epsilon k A^2\right) \operatorname{sech}^4 \zeta + \left(4\delta c k^2 A - cA\right) \operatorname{sech}^2 \zeta = 0 \tag{15}$$

and solving Eq. (15) for nonzero $\operatorname{sech}^4 \zeta$ and $\operatorname{sech}^2 \zeta$ gives

$$A = \mp 3c \frac{\sqrt{\delta}}{\epsilon}, \qquad k = \mp \frac{1}{2} \sqrt{\frac{1}{\delta}} \tag{16}$$

Thus the bright soliton solution is formed as

$$u(x,t) = A \operatorname{sech}^2 \left(\frac{\bar{k} x^\beta}{\Gamma(1+\beta)} - \frac{\bar{c} t^\beta}{\Gamma(1+\beta)} \right) \tag{17}$$

where A is given in (16). The simulations of motion of bright solitons for various values of β are demonstrated in Fig. 1(a)–(d) for $\delta = 1$, $\epsilon = 3$ and $c = 1$. When β is smaller such as $\beta = 0.25$, the shape of known single positive solitary is not clear. Increasing β to 0.50 makes the sides of the wave sharper when compared with the case $\beta = 0.25$. When $\beta = 0.75$, the shape is more close to a positive single solitary wave. In the final case, the sides of the wave sharp, the shape is narrower for $\beta = 1$. It should also be pointed out that the velocity of the wave decreases when β increases.

3.2. Singular solution

Let

$$U(\zeta) = A \operatorname{csch}^p \zeta, \quad \zeta = \frac{\bar{k} x^\beta}{\Gamma(1+\beta)} - \frac{\bar{c} t^\beta}{\Gamma(1+\beta)} \tag{18}$$

be a solution for the Eq. (12) with constants A, $\bar{k}$ and $\bar{c}$. Since the solution has to satisfy the Eq. (12), substituting it into the equation gives

$$\left(\delta p c k^2 A + \delta p^2 c k^2 A\right) \operatorname{csch}^{p+2} \zeta + \left(\delta p^2 c k^2 A - Ac\right) \operatorname{csch}^p \zeta + \epsilon k A^2 \operatorname{csch}^{2p} \zeta = 0 \tag{19}$$

Choosing $p + 2 = 2p$ gives $p = 2$ and reduces (19) to

$$\left(6\delta c k^2 A + \epsilon k A^2\right) \operatorname{csch}^4 \zeta + \left(4\delta c k^2 A - Ac\right) \operatorname{csch}^2 \zeta = 0 \tag{20}$$

Solution of (20) for nonzero csch function give

$$A = \pm \frac{3c\sqrt{\delta}}{\epsilon} \qquad k = \pm \frac{1}{2} \sqrt{\frac{1}{\delta}} \tag{21}$$

Thus the singular solution of Eq. (12) becomes

$$U(\zeta) = A \operatorname{csch}^p \frac{\bar{k} x^\beta}{\Gamma(1+\beta)} - \frac{\bar{c} t^\beta}{\Gamma(1+\beta)} \tag{22}$$