Chaos, Solitons and Fractals 96 (2017) 154–159

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Frontiers

Structure of the correlation function at the accumulation points of the logistic map

K. Karamanos [a], I.S. Mistakidis [b], S.I. Mistakidis [c,*]

[a] Complex Systems Group, Institute of Nuclear and Particle Physics, NCSR Demokritos, Aghia Paraskevi Attiki, GR 15310, Greece
[b] Hellenic Army Academy, Section of Physical Sciences and Applications, Vari Attiki, Greece
[c] Zentrum für Optische Quantentechnologien, Universität Hamburg, Luruper Chaussee 149, Hamburg 22761, Germany

ARTICLE INFO

Article history:
Received 10 July 2016
Revised 11 December 2016
Accepted 30 January 2017
Available online 7 February 2017

Keywords:
Correlation function
Symbolic dynamics
Bifurcation points
Feigenbaum attractors
Logistic map

ABSTRACT

The correlation function of the trajectory exactly at the Feigenbaum point of the logistic map is investigated and checked by numerical experiments. Taking advantage of recent closed analytical results on the symbol-to-symbol correlation function of the generating partition, we are in position to justify the deep algorithmic structure of the correlation function apart from numerical constants. A generalization is given for arbitrary $m \cdot 2^{\infty}$ Feigenbaum attractors.

1. Introduction

Recently, the study of Complex Systems has gained significant attention. One of the basic aspects of this progress is related with the understanding of correlations in and between such complex systems, which is realized through the use of different complexity measures. Among these, one can mention the transinformation [1–3], the block entropies [4–11] different types of correlation functions [12–17] and number-theoretic notions [6,18].

One of the Paradigms of Complex Systems is the logistic map. The logistic map has a simple definition but presents complex behavior when fine tuning the control parameter values. In particular, after Feigenbaum's work, the period-doubling route to chaos has been fairly understood. Also, connections with the theory of second order phase transitions (critical phenomena) have been established and scaling relations have been reported nearby the accumulation point (also called Feigenbaum Point (FP)) with and without the presence of external noise. Furthermore, cantorian fractal structures have been revealed in the transition point connecting the physics of the non-chaotic attractor with self-similarity [19–25]. Recently also, a direct connection with Experimental Mathematics has been established, too [26].

On the other hand, in Non-linear physics, the importance of the study of the correlation function has been realized from the very beginning. Particularly inspiring have been the works of Ruelle [27], Daems and Nicolis [28], and Alonso et al. [12], for the case of resonances of chaotic dynamical systems. In addition, based on the analogies between the period doubling transition and critical phenomena, Schuster has done a guess on the functional form of the correlation function of the trajectory [17]. Indeed, according to his arguments the correlation function should follow a power law behavior. In contrast, here, we demonstrate that the correlation function possesses a stratified structure. More recently, using the Feigenbaum renormalization group transformation it has been shown [29] that the correlation function of the trajectory in the one dimensional nonlinear dissipative logistic map is made of a family of power laws with a common scaling factor given by the Feigenbaum constant α. In the present work in order to extract the form of the correlation function of the trajectory we propose some more elaborated arguments, using a different approach which is based on the structure of the symbol-to-symbol correlation function [9], that is the correlation function of symbolic dynamics.

After establishing rigorously in a previous work [9] the detailed form of the symbol-to-symbol correlation function we turn now our attention to the structure of the correlation function of the trajectory. To be more concrete, taking advantage from the analytic form of the symbol-to symbol correlation function and presenting simple arguments we shall show that one can extract up to a good approximation, that is apart from numerical constants, the detailed structure for the correlation function of the trajectory. The above investigation is mainly supported by a detailed numerical

* Corresponding author. Tel.: +4917672488946.
E-mail address: smistaki@physnet.uni-hamburg.de (S.I. Mistakidis).

http://dx.doi.org/10.1016/j.chaos.2017.01.020
0960-0779/

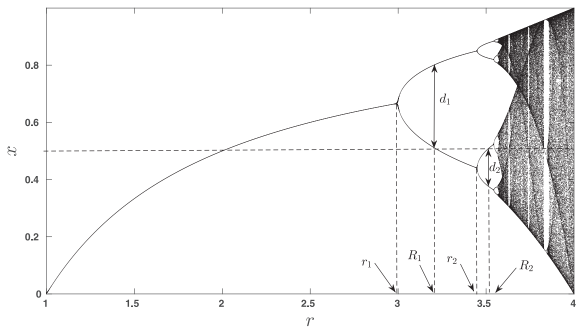

Fig. 1. The bifurcation diagram for the logistic map for the superstable 2^n-cycles. It is shown the control parameter values r_i for the first few bifurcation points and the values R_i for the superstable orbits.

study which takes into account a large enough statistical sample of the logistic map. In this manner, we can justify the analytic form of the correlation function of the trajectory from first principles using the Metropolis-Stein and Stein algorithm (MSS algorithm), apart from numerical constants, which depend on the detailed functional form of the map. Furthermore, we make an attempt to generalize these results for an arbitrary $m \cdot 2^\infty$ accumulation point [30], for $m = 2, 3, \ldots,$ which correspond to the accumulation points of the bifurcation tree [17,31] (see also Fig. 1). Finally, a general form for the correlation function of the trajectory and that obtained from the symbolic dynamics is also suggested. We believe that our results will inspire similar investigations on non-unimodal maps and give further insight providing new complexity measures on real experimental time-series.

The paper is organized as follows. In Section 2 we introduce the logistic map and the definitions of different types of correlation functions that will be used. In Section 3 we present our careful numerical experimentation for the symbol-to-symbol correlation function and for the correlation function of the trajectory at the (first) accumulation point. As it is shown those functions satisfy simple numerical prescriptions, which are explicitly outlined. In addition, we propose some simple arguments which, up to a good approximation, allow for the justification of the functional form of the correlation function of the trajectory from the symbol-to-symbol correlation function apart from arithmetical constants in a systematic basis. We then present analogous results and generalizations for the $m \cdot 2^\infty$ accumulation points. Finally, in Section 4 we draw the main conclusions and discuss future plans.

2. The logistic map

The logistic map is the archetype of a Complex System. Let us elaborate. We introduce the logistic map in its familiar form

$$x_{n+1} = r x_n (1 - x_n), \tag{1}$$

where r is the control parameter value and n denotes the respective iteration of the map. For the logistic map in this form the generating partition is easily computed, following an argument dating back to the French Mathematician Gaston Julia. To be more specific, for $f(x) = rx(1-x)$ the equation $f'(c) = 0$ gives c = 0.5, so that the partition of the phase space (which in this case coincides with the unit interval I = [0,1]) L = [0,0.5] and R = (0.5,1] is a generating one (see also [32] for a more rigorous definition). Notice that according to Metropolis et al. [33] the information content of the symbolic trajectory is the "minimum distinguishing information". Needless to say, in this representation the logistic map can be viewed as an abstract information generator.

In particular, the period doubling route to chaos has been fairly studied and it is by now well understood. These studies led to the occurrence of the two Feigenbaum constants α and δ which can be defined by an approximate real space renormalization procedure. Especially, the constant δ is related with the spacing in the control parameter space of the successive values of occurrence of the superstable periodic orbits and can be roughly estimated by the bifurcation diagram [22,23]. If we denote as $\{R_n\}$ this set of values, δ is defined as

$$\delta = \lim_{n\to\infty} \frac{R_n - R_{n-1}}{R_{n+1} - R_n}, \tag{2}$$

and for the quadratic map reads

$$\delta \simeq 4.669201609102990\ldots. \tag{3}$$

Moreover, the constant α is related to the rescaling of the period doubling functional composition law and its value for the logistic map reads

$$\alpha = -\lim_{n\to\infty} \frac{d_n}{d_{n+1}} \simeq -2.5029078750095892\ldots. \tag{4}$$

Finally, note that the constants α, δ are related as it can be shown by using renormalization group arguments (see [16,34] and references therein). The values of the above two constants depend only on the order of the maximum and have long been studied. They are thus, for instance, universal for quadratic maps irrespectively of the exact way one writes down the map.

Fig. 1 presents the control parameter values of the bifurcation points denoted as r_1, r_2, r_3, ... while the corresponding values for the superstable orbits are depicted as R_1, R_2, R_3, ... The values of d_i figuring in the definition of the Feigenbaum constant α are also shown. Note here that Feigenbaum and successors have shown that Eq. (2), holds if instead of R_i we use r_i.

After the above brief introduction of the logistic map and its properties, we shall next define the (un-normalized) correlation function of the trajectory as

$$C_{un}(m) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} x_{i+m} x_i, \tag{5}$$

where the deviation from the real value of the map at the ith iteration is given by $x_i = f^i(x_0) - \bar{x}$ and the corresponding mean value of the map taking into account N iterations (sample) is denoted by $\bar{x} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f^i(x_0)$. Also, in direct analogy with the above defined un-normalized correlation function one can also introduce here the normalized correlation function

$$C(m) = \frac{C_{un}(m)}{C_{un}(0)} = \frac{C_{un}(m)}{\sigma^2}, \tag{6}$$

where σ is the mean standard deviation, which normalizes the statistical data.

From the above definitions follows that $C(m)$ (or equally $C_{un}(m)$) yields another measure for the irregularity of the sequence of iterates x_0, $f(x_0)$, $f^2(x_0)$, ... etc. It tells us how much the deviations of the iterates from their average value, $x_i = x_i - \bar{x}$ that are m steps apart (i.e. x_{i+m} and x_i) "know" about each other, on the average. Another remark here is that if $C(m) \nrightarrow 0$ as $m \to \infty$ then the system does not have the mixing property.

We should here note that the problem of determining the correlation function of an arbitrary dynamical system is difficult to calculate in the general case. This is the reason to resort to other computable observables such as the symbol-to-symbol correlation function [28]. Thus, in direct analogy with the correlation function of the trajectory one can introduce the un-normalized symbol-to-symbol correlation function as

$$K_{un}(m) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} y_{i+m} y_i, \tag{7}$$