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Systematic evaluation of a new combinatorial curvature for complex networks

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ABSTRACT

We have recently introduced Forman's discretization of Ricci curvature to the realm of complex networks. Forman curvature is an edge-based measure whose mathematical definition elegantly encapsulates the weights of nodes and edges in a complex network. In this contribution, we perform a comparative analysis of Forman curvature with other edge-based measures such as edge betweenness, embeddedness and dispersion in diverse model and real networks. We find that Forman curvature in comparison to embeddedness or dispersion is a better indicator of the importance of an edge for the large-scale connectivity of complex networks. Based on the definition of the Forman curvature of edges, there are two natural ways to define the Forman curvature of nodes in a network. In this contribution, we also examine these two possible definitions of Forman curvature of nodes in diverse model and real networks. Based on our empirical analysis, we find that in practice the unnormalized definition of the Forman curvature of nodes with the choice of combinatorial node weights is a better indicator of the importance of nodes in complex networks.

1. Introduction

Discrete mathematics, especially graph theory [1,2], has made vast contributions towards our understanding of the structure of complex networks [3–9]. In network theory [3–9], a recent focus has been the development of measures inspired by geometry [10–22] for the characterization of complex networks. Curvature is a central concept in geometry, and in particular, Ricci curvature is a classical notion of Riemannia n geometry that quantifies both volume growth of infinitesimal balls as well as dispersion rate of geodesics [23]. Two discretizations of the classical Ricci curvature, Ollivier–Ricci curvature [24–27] and Forman–Ricci curvature [28], have been introduced to the domain of complex networks [11,12,14–17,19]. Firstly, Ollivier [24–27] in 2007 proposed a discretization of the Ricci curvature. Subsequently, Ollivier–Ricci curvature was adapted to the setting of undirected graphs, and this concept has proven to be successful in the analysis of complex networks [11,12,14–17]. Secondly, even before Ollivier, Forman [28] had devised another discretization of the Ricci curvature. Recently, we [19] have adapted the Forman–Ricci curvature to the domain of complex networks.

In the context of complex networks, the Forman curvature captures the second property of the Ricci curvature, namely, the dispersion rate of geodesics. This is achieved by adapting to the discrete setting of graphs, the classical Bochner–Weitzenböck formula [23], which gives the connection between curvature and the Laplacian on a Riemannian manifold. The resulting definition of the Forman curvature of an edge [19] is remarkably simple to compute in complex networks. Importantly, the definition of the Forman curvature [19] captures on the one hand the combinatorial properties of the network and on the other hand naturally incorporates the weights of nodes and edges in the network. Thus, Forman curvature is suitable for the geometrical characterization of both weighted and unweighted networks. In our recent work [19], we have successfully shown that Forman curvature represents a natural, as well as computationally efficient tool for the analysis and classification of complex networks.

In contrast to node-based measures such as degree, clustering coefficient [4,29] and betweenness centrality [8,30], Forman curvature [19] is an edge-based measure that quantifies the extent of spreading at the ends of edges in a network. The more is the spreading at the ends of edges, the more negative is the Forman

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curvature. We [19] have previously shown that the Forman curvature is an indicator of the relative importance of edges in model and real networks. In contrast to several node-based measures, relatively few edge-based measures have been proposed to date for the analysis of complex networks. Besides Forman curvature [19], the available edge-based measures include edge betweenness [30,31], embeddedness [32] and dispersion [33]. Edge betweenness [30,31], embeddedness [32] and dispersion [33] have also been employed to quantify the importance of edges in complex networks, in particular, social networks. In this contribution, we therefore perform a comparative analysis of Forman curavture with edge betweenness, embeddedness and dispersion in model and real networks.

An attractive feature of the Forman curvature is its applicability to networks with any set of positive weights for nodes and edges [19]. This feature renders the Forman curvature a powerful tool for analysis of weighted networks where the user can specify the best set of weights for nodes and edges. However, the available maps of real networks seldom specify the weights of nodes or edges in the network. While analyzing networks without prescribed weights, there are at least two possible choices for the node weights in the definition of the Forman curvature of an edge which are introduced in the next section. In this contribution, we also study in model and real networks the effect of the two possible choices of node weights in the definition of the Forman curvature of an edge.

Although Forman curvature is a measure associated to edges, it is also natural to desire an extension of the concept to nodes as most measures employed in network theory are associated to nodes in a network. However, intrinsically there isn't a unique manner to define the Forman curvature of a node based on the definition of the Forman curvature of an edge in a network. Instead, two natural ways present themselves to define the Forman curvature of a node based on the definition of the Forman curvature of an edge which are introduced in the next section. In this contribution, we also systematically examine in model and real networks the two definitions of the Forman curvature for nodes.

The remainder of this paper is organized as follows. In Section 2, we present a brief review of the Forman curvature, the definition of the Forman curvature of an edge, the two choices of node weights in the definition of the Forman curvature of an edge, and the two possible definitions of the Forman curvature of a node in a network. In Section 3, we describe the model and real networks investigated here. In Section 4, we present our results, and in Section 5, we conclude with a short summary and possible applications.

2. Theory

2.1. Forman curvature – a brief overview

Forman's [28] discretization of the classical Ricci curvature is applicable to a large class of geometric objects, the so called (*regular*) *CW complexes*. Forman's discretization of the Ricci curvature [28] is derived on the basis of the so called Bochner–Weitzenböck formula [23] which in its classical form connects curvature and the Laplacian on a Riemannian manifold. However, while the well known version of the Bochner–Weitzenböck formula relates the Laplacain operator of functions defined on a given manifold, a relatively less known though by no means any less important version of the Bochner–Weitzenböck formula concerns *forms* rather than functions. It is this less well known variant of the Bochner–Weitzenböck formula that enables adaptation to the large class of geometric objects, the so called *CW* complexes, where cells play the role of the forms in the original, standard setting.

It is important to underline that the definition of the Forman curvature is valid for a wide spectrum of possible weights. In his original contribution, Forman [28] had motivated the weights in his curvature definition from the point of view of simple geometrical quantities such as length, area and volume. However, the spectrum of possible sets of weights that are admissible in the definition of Forman curvature is far more extensive and general. An important if not the main motivation behind considering weighted manifolds (or, as in Forman's work, CW complexes), stems from the observation made by Cheeger, Gromov and others [34,35] that to control collapse (degeneracy) of manifolds under curvature bounds (mainly, Ricci curvature bounds), one has to consider the volume and also more general measures. For other motivations, such as appertaining to minimal surfaces, we refer the reader to [36]. While in several approaches, the considered measures typically satisfy certain smoothness properties, Forman's method allows for weights that do not satisfy any such properties. Indeed, Forman's approach can be viewed as an extreme discretization of the notion of metric measure space, where the underlying structure is *well behaved* (a manifold), and the overlay measure also satisfies similar properties, by replacing the smooth manifolds with the more general *CW* complexes along with not imposing any constraints on the attached discrete measure (i.e., the given weights). That being said, one of the remarkable characteristics of Forman's work is the fact that a geometric structure, so to say, can be recovered from any set of weights. More precisely, any given set of (positive) weights involved in the computation of the Ricci curvature, can be arbitrarily well approximated by a set of *natural* or *geometric* weights, i.e., weights that conform to dimensional scaling properties similar to those of the truly geometric measures such as length, area and volume.

2.2. Forman curvature of an edge

As classical Ricci curvature operates directionally along the vectors, the concept is naturally defined for edges in the network. Although for the general n-dimensional case, the Bochner–Weitzenböck formula and the curvature term is given by a quite complicated formula, in the limiting 1-dimensional case of graphs or networks, the mathematical formula for the Forman curvature of an edge e in the network, as given by [19], is quite simple:

$$\mathbf{F}(e) = w_e\left(\frac{w_{v_1}}{w_e} + \frac{w_{v_2}}{w_e} - \sum_{e_{v_1}\,\sim\, e,\ e_{v_2}\,\sim\, e}\left[\frac{w_{v_1}}{\sqrt{w_e w_{e_{v_1}}}} + \frac{w_{v_2}}{\sqrt{w_e w_{e_{v_2}}}}\right]\right); \tag{1}$$

where e denotes the edge under consideration between two nodes v_1 and v_2, w_e denotes the weight of the edge e under consideration, w_{v_1} and w_{v_2} denote the weights associated with the nodes v_1 and v_2, respectively, $e_{v_1} \sim e$ and $e_{v_2} \sim e$ denote the set of edges incident on nodes v_1 and v_2, respectively, after excluding the edge e under consideration which connects the two nodes v_1 and v_2. Note that the indices $e_{v_1} \sim e$ and $e_{v_2} \sim e$ below the summation sign on the right hand side of Eq. (1) do not specify a double sum but rather specify a single sum, that is,

$$\sum_{e_{v_1}\,\sim\, e,\ e_{v_2}\,\sim\, e}\left[\frac{w_{v_1}}{\sqrt{w_e w_{e_{v_1}}}} + \frac{w_{v_2}}{\sqrt{w_e w_{e_{v_2}}}}\right] = \sum_{e_{v_1}\,\sim\, e}\frac{w_{v_1}}{\sqrt{w_e w_{e_{v_1}}}} + \sum_{e_{v_2}\,\sim\, e}\frac{w_{v_2}}{\sqrt{w_e w_{e_{v_2}}}}$$

As we have have emphasized before, any collection of positive weights can be inputed in the formula for the Forman curvature of an edge, a proof of the flexibility and adaptiveness of Forman's curvature. Here, we would like to bring to the readers' attention the fact that, in practice, one might be confronted with two sets