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# Restricted fractional differential transform for solving irrational order fractional differential equations

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ABSTRACT

Arikoglu and Ozkol developed a new semi-analytical numerical technique, fractional differential transform method (FDTM), for solving fractional differential equations (FDEs). FDTM was not achieved for solving irrational order fractional differential equations. Here we develop a new method to be applicable for solving rational or irrational order FDEs. This method is called the restricted fractional differential transform method (RFDTM). In fact, RFDTM is based on the restriction of the classical two dimensional differential transform methods. A useful theorem is provided, and Several FDEs are solved by using RFDTM. Moreover, several illustrative examples are presented to demonstrate the accuracy and effectiveness of the proposed method.



## 1. Introduction

Fractional calculus and FDEs are widely explored subject due to the great importance of many applications in fluid mechanics, biology, control theory of dynamical systems, probability and statistics, viscoelasticity, polymer modeling, finance, physics and engineering, see e.g. Schneider and Wyss [1], Mainardi [2], Magin et al. [3], Magin [4], Metzler and Klafter [5], Beyer and Kempfle [6], Lederman et al. [7], Bagley and Torvik [8], Riewe [9], Kulish and Lage [10], Wyss [11], Song and Wang [12], and the works by Diethelm and Freed cf. Keil et al. [13]. Comprehensive reviews of literature concerning the application of fractional differential equations may be found in the books by Oldham and Spanier [14], Diethelm [15], and Podlubny [16].

There are very few of FDEs can be solved analytically. Thus, accurate and efficient numerical techniques are needed. Various semi-numerical techniques have been proposed for approximate solutions of the fractional order differential and fractional order integral equations, For example, the Adomian decomposition method [17–21], variational iteration method [18,22–26], homotopy perturbation method [27–30], Wavelet Method [31,32], fractional difference method [17,33], and extrapolation method [34]. Another efficient and accurate semi-numerical method, such as FDTM, was introduced by Arikoglu and Ozkol [35] to solve linear and nonlinear initial value problems of fractional order, which utilize the form of fractional power series as the approximation to the exact solution. It is appropriate to note that FDTM is currently of considerable interest for solving FDEs, see, e.g. ErtÄurk and Momani [36], Oturanc et al. [37], Arikoglu and Ozkol [38], and later work followed by Nazari and Shahmorad [39].

The motivation for the present article is the work of Arikoglu and Ozkol [35,38]. Arikoglu and Ozkol [35] developed the differential transform method (DTM) to introduce a new analytical technique for solving FDEs that is named as FDTM. They argued that one should expand the analytic solution function as the following fractional power series [40],

$$y(x)=\sum_{k=0}^{\infty}F(k)(x-x_0)^{\frac{k}{\gamma}} \tag{1}$$

where $\gamma$ is the order of the fraction to be selected and $F(k)$ is the kth fractional differential transform of $y(x)$given by

$$F(k)=\begin{cases}\dfrac{1}{(k/\gamma)!}\left.\dfrac{d^{k/\gamma}f(x)}{dx^{k/\gamma}}\right|_{x=x_0} & k/\gamma\in Z^{+}\\ 0 & k/\gamma\notin Z^{+}\end{cases} \tag{2}$$

Now, if the considering FDEs include a fractional derivative, say$D^{\alpha}$, where $\alpha$ is irrational number, then one cannot find $\gamma$ in Eqs. (1) and (2). So that, it is impossible to apply FDTM for solving FDEs. The object of this work is to present a new technique, RFDTM, to be applicable to solve FDEs, which including rational or irrational fractional derivative. In the following sections RFDTM will be presented.

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## 2. Restricted fractional differential transform method

There are many definitions of fractional order derivative, e.g. Riemann–Liouville derivative, Grünwald–Letnikov derivative, Caputo derivative, Sonin–Letnikov derivative, Miller–Ross derivative, Hadamard derivative, Weyl derivative, Marchaud derivative, Riesz–Miller derivative, Erdélyi–Kober derivative [15,16,41]. Caputo derivative is always used in FDEs to express of many real world physical problems since it has the advantage of defining integer order initial conditions. The Caputo derivative for any analytic function $u(x)$ is defined by [40]

$$^{C}D^{\alpha}_{x_0}u(x) = \frac{1}{\Gamma(n-\alpha)}\int_{x_0}^{x}\frac{u^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, \quad n-1<\alpha<n,\ n\in Z^{+} \tag{3}$$

Let $f(x, y)$: $R^2 \to R$ be analytical function then it can be express as multi Taylor series about $(x_0, y_0)$as follows:

$$f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!\,j!}\left(\frac{\partial^{i+j}f(x,y)}{\partial x^i\partial y^j}\right)_{(x_0,y_0)}(x-x_0)^i(y-y_0)^j \tag{4}$$

By setting

$$F(i,j) = \frac{1}{i!\,j!}\left(\frac{\partial^{i+j}f(x,y)}{\partial x^i\partial y^j}\right)_{(x_0,y_0)} \tag{5}$$

Then

$$f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}F(i,j)(x-x_0)^i(y-y_0)^j \tag{6}$$

Clearly, $F(i, j)$ in Eq. (5) is the two dimensions differential transform of the function$f(x, y)$, while Eq. (6) represent the differential inverse transform of $F(i, j)$.

Now, If $u(x) = f(x,y)|_{y=(x-x_0)^{\alpha}+y_0}$, where $\alpha > 0$, that is, the two dimensional function $f(x, y)$ is restricted to one dimensional function $u(x)$, then Eq. (5) and Eq. (6) respectively, become

$$U(i,j) = F(i,j) = \frac{1}{i!\,j!}\left(\frac{\partial^{i+j}f(x,y)}{\partial x^i\partial y^j}\right)_{(x_0,y_0)} \tag{7}$$

$$u(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U(i,j)(x-x_0)^{i+\alpha j} \tag{8}$$

Eq. (7) is called the restricted fractional differential transform (RFDTM), while Eq. (8) is called inverse of RFDTM.

Now, let $u(x)$, $v(x)$ and $w(x)$ can be express as $u(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U(i,j)x^{i+\alpha j}$, $v(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}V(i,j)x^{i+\alpha j}$ and $w(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}W(i,j)x^{i+\alpha j}$ respectively, then the fundamental mathematical operations performed by RFDTM are introduced in the following theorems.

**Theorem (1).** *If* $w(x) = u(x) + v(x)$ *then* $W(i,j) = U(i,j) + V(i,j)$, for $i \ge 0$ , $j \ge 0$.

**Proof** ((1))**.** It hold directly.

**Theorem (2).** *If* $w(x) = u(x)v(x)$ *then* $W(i,j) = \sum_{k=0}^{j}\sum_{r=0}^{i}U(r,j-k)\,V(i-r,k)$ for $i \ge 0$, $j \ge 0$.

**Proof** ((2))**.**

$$w(x) = u(x)v(x)$$

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}W(i,j)x^{i+\alpha j} = \left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}U(i,j)x^{i+\alpha j}\right)\times\left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}V(i,j)x^{i+\alpha j}\right) = \left(\sum_{j=0}^{\infty}\beta_j x^{\alpha j}\right)\left(\sum_{j=0}^{\infty}\gamma_j x^{\alpha j}\right)$$

where $\gamma_j = \sum_{i=0}^{\infty}U(i,j)x^i$, $\beta_j = \sum_{i=0}^{\infty}V(i,j)x^i$

$$\left(\sum_{j=0}^{\infty}\beta_j x^{\alpha j}\right)\left(\sum_{j=0}^{\infty}\gamma_j x^{\alpha j}\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\beta_k\gamma_{j-k}x^{\alpha j}$$

$$= \sum_{j=0}^{\infty}\omega_j x^{\alpha j}, \quad \omega_j = \sum_{k=0}^{j}\beta_k\gamma_{j-k}$$

$$\omega_j = \sum_{k=0}^{j}\left(\sum_{i=0}^{\infty}U(i,k)x^i\right)\left(\sum_{i=0}^{\infty}V(i,j-k)x^i\right)$$

$$= \sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{r=0}^{i}U(i,k)V(i-r,j-k)x^i$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{i}U(i,k)V(i-r,j-k)x^i$$

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}W(i,j)x^{i+\alpha j} = \sum_{j=0}^{\infty}\omega_j x^{\alpha j}$$

$$= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{i}U(i,k)V(i-r,j-k)x^{i+\alpha j},$$

By comparing the coefficients of $x$, one can have

$$W(i,j) = \sum_{k=0}^{j}\sum_{r=0}^{i}U(i,k)V(i-r,j-k)$$

**Theorem (3).** *If* $v(x) = x^{m+\alpha n}u(x)$, *where m and n are an integer number then*

$V(i,j) = 0$ for $i < m$ *or* $j < n$
$V(i,j) = U(i-m,j-n)$ for $i \ge m$ *and* $j \ge n$

**Proof (3).**

$$v(x) = x^{m+\alpha n}u(x),$$

$$x^{m+\alpha n}u(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U(i,j)\,x^{i+m+\alpha j+\alpha n}$$

$$v(x) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}V(i,j)\,x^{i+\alpha j} + \sum_{i=0}^{m-1}\sum_{j=n}^{\infty}V(i,j)\,x^{i+\alpha j} + \sum_{i=m}^{\infty}\sum_{j=0}^{n-1}V(i,j)\,x^{i+\alpha j} + \sum_{i=m}^{\infty}\sum_{j=n}^{\infty}V(i,j)\,x^{i+\alpha j}$$

$$= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}V(i,j)\,x^{i+\alpha j} + \sum_{i=0}^{m-1}\sum_{j=n}^{\infty}V(i,j)\,x^{i+\alpha j} + \sum_{i=m}^{\infty}\sum_{j=0}^{n-1}V(i,j)\,x^{i+\alpha j} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}V(i+m,j+n)\,x^{i+m+\alpha j+\alpha n}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U(i,j)\,x^{i+m+\alpha j+\alpha n}$$