Chaos, Solitons and Fractals 94 (2017) 37–43

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

About the continuity of reachable sets of restricted affine control systems

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ARTICLE INFO

Article history:
Received 30 August 2016
Revised 11 November 2016
Accepted 14 November 2016
Available online 26 November 2016

MSC:
93B03
93B99
93C15

Keywords:
Affine system
Accessible sets
Lower semi-continuity
Hausdorff metric

ABSTRACT

In this paper we prove that for a restricted affine control system on a connected manifold M, the associated reachable sets up to the time t varies continuously in each independent variable: time, state and the range of the admissible control functions. However, as a global map it is just lower semi-continuous. We show a bilinear control system on the plane where the global map has a discontinuity point. According to the Pontryagin Maximum Principal, in order to synthesizes the optimal control the Hausdorff metric continuity is crucial. We mention some references with concrete applications. Finally, we apply the result to the class of Linear control systems on Lie groups.

1. Introduction

A control system $\Sigma = (M, \mathcal{D})$ is determined by a manifold M and a family of differential equations $\mathcal{D}$ induced by a class of admissible control functions. For $x \in M$ the accessible set of Σ from x, i.e., the set of points that can be reached from x through all possible $\mathcal{D}$-trajectories in positive time, have been investigated in several works from different points of view. For instance, a control system has the accessibility property from x if the reachable set from x , has non-empty interior in the M topology, [25,42] . The description of this class of sets have been analyzed by, Darken [21] , Gronski [23], Lobry [34] and Sussmann and Jurdjevic [41]. Also, in [30,31] the author makes an effort to describe the structure of these sets for special systems on low dimension. Actually, the accessible sets are difficult to describe because they are boundary points that can only be reached by chattering controls, i.e., infinite number of switched of controls in finite time.

From a particular state $x \in M$, the *controllability property of* Σ means that starting from x it is possible to reach any point of the space state by using the available controls in positive time. In other words, *the reachable set fromxmust be the wholeM*. The study of controllability has been a subject of huge interest and has generated an enormous activity in research for different classes of control system. Specially, on Linear and Bilinear systems on Euclidean spaces, [20,24,43]. And Linear and Invariant systems on Lie groups. For linear systems we mention [1–9,12–14,16] and [27]. For invariant we refer to the father of this class of systems [19], and [38] and a complete list of references therein.

Furthermore, for a restricted admissible class of control $\mathcal{U}$, in [20] the authors introduce the notion of *control set,*a subset $\mathcal{C}$ of M where controllability holds at the interior $int(\mathcal{C})$ of $\mathcal{C}$ and approximately controllable at the boundary $\partial\mathcal{C}$ of $\mathcal{C}$. Then, they prove that the map

$$\mathcal{U}(\rho) \to \ \rho\text{-control set}$$

is lower semi-continuous. Here, $\rho > 0$ is a parameter which allows to increase (respect to $\subset$) the admissible class of control function $\mathcal{U}$ by increment the range of the controls. See also, [17,36].

On the other hand, in his book [35], Pontryagin shows that for a restricted classical linear control system on Euclidean spaces, the accessible set up to the positive time t is compact, convex and having the form changed continuously on time with the Hausdorff metric. The Pontryagin Maximum Principal is a very powerful theorem for concrete applications in a broad spectrum of disciplines.

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[1] Supported by Proyecto Fondecyt no. 1150292, Conicyt, Chile
[2] Supported by Proyecto Fondecyt no. 1151159, Conicyt, Chile
[3] Supported by Fapesp grant no. 2016/11135-2.

http://dx.doi.org/10.1016/j.chaos.2016.11.006
0960-0779/

For instance, for application in mechanics see [28], in control of rail vehicles [32], in aerospace systems [33,40], in economy, [39], etc.

In our particular case, given an initial condition x and an arbitrary by fix compact and convex subset Ω of $\mathbb{R}^m$, the continuity of the application

$$\mathcal{R}_{x,\Omega} : t \mapsto \mathcal{R}_{\leq t,\Omega}(x) \subset M$$

is crucial in the proof of the celebrated Lenin Price Pontryagin Theorem. Actually, in the classical optimal time for a Linear Control System on $\mathbb{R}^n$, the continuity of $\mathcal{R}_{x,\Omega}$ allows to build the optimal control. In fact, in this particular case, $\mathcal{R}_{\leq t,\Omega}(x)$ is also convex and if t^* is the optimal time associated to the optimal control u^*, then the ending point of the optimal curve $\varphi(t, x, u^*)$, i.e., the point $\varphi(t^*, x, u^*)$, must belong to the boundary of $\mathcal{R}_{\leq t^*,\Omega}(x)$, otherwise is interior! By applying the Banach Theorem, there exists a hyperplane H_{t^*} which leave the whole reachable set in one side of H_{t^*}. It turns out that there exists a covector η_{t^*} orthogonal to H_{t^*} such that

$$< \eta_{t^*}, z > \quad \leq 0 \text{ for any } z \in \mathcal{R}_{\leq t^*,\Omega}(x)$$

and the maximum equals to zero is attainable exactly on the boundary point $\varphi(t^*, x, u^*)$. By the Bellman Maximum Principle, any point of the curve must be optimal. Hence, the existence of a 1-parameter curve of covectors follows, which is the main ingredient of the PMP to synthesize the optimal control and solve the problem.

Our work is the first attempt to prove a similar result for the class of Linear Control Systems on Lie Groups introduced in [12]. In this article we just take care of the Hausdorff continuity part. But, for a more general class of systems. In the near future we expect to analyze convexity through some notion of geodesic of the system and to try to get the same Pontryagin result for linear system on Lie groups.

Precisely, consider a restricted affine control system on a connected Riemannian $\mathcal{C}^\infty$-manifold M, determined by the family of differential equations

$$\Sigma_\Omega : \qquad \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \text{ with } u \in \mathcal{U}_\Omega.$$

Where

$$\mathcal{U}_\Omega = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m); \quad u(t) \in \Omega\}$$

is the class of restricted admissible control functions with Ω being a compact and convex subset of $\mathbb{R}^m$ with $0 \in int(\Omega)$.

If $x \in M$ and $u \in \mathcal{U}_\Omega$, we denote by $\varphi(t, x, u)$ the Σ_Ω-solution satisfying $\varphi(0, x, u) = x$. The reachable set $\mathcal{R}_{\leq t,\Omega}(x)$ of Σ_Ω is built with the points of M which are possible to reach starting from the initial condition x, through all Σ_Ω-solutions in nonnegative time less or equal than t.

It is well known that the map

$$(t, x, u) \in \mathbb{R} \times M \times \mathcal{U}_\Omega \mapsto \varphi(t, x, u) \in M$$

is continuous. Furthermore, the set $\mathcal{U}_\Omega$ is a compact metrizable space in the weak* topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^*$ (see for example [29]). As usual V^* denotes the dual of the vector space V.

In this paper we give a direct proof that for a restricted affine control system Σ_Ω on a connected manifold M, the associated reachable sets up to time t varies continuously on each variable separately by fixing the others. Precisely, the maps

$$t \mapsto \mathcal{R}_{\leq t,\Omega}(x), \quad x \mapsto \mathcal{R}_{\leq t,\Omega}(x) \quad \text{and} \quad \Omega \mapsto \mathcal{R}_{\leq t,\Omega}(x)$$

are continuous.

The variable Ω belongs to the metric space $(\mathrm{Co}(\mathbb{R}^m), d_H)$ where

$$\mathrm{Co}(\mathbb{R}^m) = \{\Omega \subset \mathbb{R}^m; \ \Omega \text{ is a non-empty compact convex subset}\}$$

and d_H is the Hausdorff metric. Moreover, $(\mathcal{C}(M), \varrho_H)$ is the metric space of all non-empty compact subsets of M with the Hausdorff metric.

As a consequence, every continuous functional J defined on the accessible set $\mathcal{R}_{\leq t,\Omega}(x)$ has a minimum and maximum at any continuity point (t, x, Ω). In fact, $J\big(\mathcal{R}_{\leq t,\Omega}(x)\big) \subset \mathbb{R}$ is compact.

The main theorem of the paper establish that the map

$$(t, x, \Omega) \in \mathbb{R} \times \mathbb{M} \times \ \mathrm{Co}(\mathbb{R}^m) \mapsto \mathcal{R}_{\leq t,\Omega}(x)$$

is lower semi-continuous.

Finally, we notice that no preliminary knowledge of control system is required to read the paper.

2. Control affine systems

Let M be a connected Riemannian $\mathcal{C}^\infty$-manifold and $f_0, f_1, \ldots, f_m \in \mathcal{X}^\infty(M)$, $m+1$ vector fields.

Definition 1. An affine control system is determined by the family of ordinary differential equations

$$\Sigma_\Omega : \qquad \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \quad \text{where} \quad u \in \mathcal{U}_\Omega.$$

The set of the control functions $\mathcal{U}_\Omega$ is defined as

$$\mathcal{U}_\Omega = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m); \quad u(t) \in \Omega\}$$

with Ω being a compact and convex subset of $\mathbb{R}^m$.

It is well known that the set of the control functions is a compact metrizable space in the weak* topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^*$, (see for instance Proposition 1.14 of [29]). As usual, V^* means the dual of the vector space V.

For a given initial state $x \in M$ and $u \in \mathcal{U}_\Omega$ we denote the solution of Σ_Ω by $\varphi(t, x, u)$. The curve $t \mapsto \varphi(t, x, u)$ is the only solution of Σ_Ω satisfying $\varphi(0, x, u) = x$ in the sense of Caratheodóry. That is, it is an absolutely continuous curve satisfying the corresponding integral equation. Throughout the paper we assume that all the solutions are defined in the whole real line. Even though this assumption is in general restrictive, there are several cases where the assumption of completeness goes without loss of generality, such as the class of linear systems on Lie groups, [15], and control affine systems on compact manifolds, [26]. Moreover, the map

$$(t, x, u) \in \mathbb{R} \times M \times \mathcal{U}_\Omega \mapsto \varphi(t, x, u) \in M$$

is a continuous map (see for instance Theorem 1.1 of [29]).

For a given state $x \in M$ and a positive time t let us introduce the sets

$$\mathcal{R}_{\leq t,\Omega}(x) = \{y \in M; \ \exists u \in \mathcal{U}_\Omega, \ s \in [0, t] \text{ with } \ y = \varphi(s, x, u)\},$$

and

$$\mathcal{R}_\Omega(x) = \bigcup_{t>0} \mathcal{R}_{\leq t,\Omega}(x).$$

$\mathcal{R}_{\leq t,\Omega}(x)$ is called the set of reachable point from x up to time t and $\mathcal{R}_\Omega(x)$ the set of reachable points from x.

Our goals include first to prove the partial continuity of the map

$$(t, x, \Omega) \stackrel{\mathcal{R}}{\to} \mathcal{R}_{\leq t,\Omega}(x).$$

Means, continuity in each variable: time t, state $x \in M$ and the range Ω of the admissible class of control $\mathcal{U}_\Omega$. Secondly, we prove that the global map $\mathcal{R}$ is lower semi-continuous.

First, we notice that it is possible to reduce the proof by considering a special class of control. In fact, let us consider the set of the piecewise control functions $\mathcal{U}_\Omega^{\mathrm{PC}} \subset \mathcal{U}_\Omega$ and define the corresponding reachable sets as

$$\mathcal{R}^{\mathrm{PC}}_{\leq t,\Omega}(x) = \left\{y \in M; \ \exists u \in \mathcal{U}_\Omega^{\mathrm{PC}}, s \in [0, t] \text{ with } \ y = \varphi(s, x, u)\right\}$$