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Chaos in hyperspaces of nonautonomous discrete systems

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ABSTRACT

We study the interaction of some dynamical properties of a nonautonomous discrete dynamical system (X, f_∞) and its induced nonautonomous discrete dynamical system $(\mathcal{K}(X), \overline{f_\infty})$, where $\mathcal{K}(X)$ is the hyperspace of non-empty compact sets in X, endowed with the Vietoris topology. We consider properties like transitivity, weakly mixing, points with dense orbit, density of periodic points, among others. We also present examples of nonautonomous discrete dynamical systems showing that transitivity, density of periodic points and sensitive dependence on initial conditions are independents on the unit interval, i.e., unlike autonomous discrete dynamical systems, in definition of Devaney chaotic there are not redundant conditions for NDS on the interval. Actually, our examples give an even more precise conclusion: the classical result stating that transitivity is a sufficient condition for an autonomous discrete dynamical system on the interval to be Devaney chaotic fails to be true for nonautonomous dynamical systems.

1. Introduction

Let X be a topological space, $f_n\colon X \to X$ a continuous function for each positive integer n, and $f_\infty = (f_1, f_2, \ldots, f_n, \ldots)$. The pair (X, f_∞) denotes the *nonautonomous discrete dynamical system* (NDS, for short) in which the *orbit of a point* $x \in X$ under f_∞ is defined as the set

$$\operatorname{orb}(x, f_\infty) = \{x, f_1(x), f_1^2(x), \ldots, f_1^n(x), \ldots\},$$

where

$$f_1^n := f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1,$$

for each positive integer n. In particular, when f_∞ is the constant sequence $(f, f, \ldots, f, \ldots)$, the pair (X, f_∞) is the usual (autonomous) discrete dynamical system given by the continuous function f on X and it will be denoted by (X, f). Autonomous dynamical systems have been studied by many authors obtaining interesting and useful results. NDS were introduced in [16], and are related to nonautonomous difference equations. Indeed, a general form of a nonautonomous difference equation is the following: Given a compact metric space (X, d) and a sequence of continuous function $(f_n : X \to X)_{n\in\mathbb{N}}$, for each $x \in X$ we set

$$\begin{cases} x_0 = x, \\ x_{n+1} = f_n(x_n). \end{cases}$$

This kind of nonautonomous difference equations has been considered by several mathematicians (see for instance, among others, [22,26]). The most classical examples are when $X = [0, 1]$ is the unit interval, and d is the usual euclidean metric. Observe that the orbit of a point forms a solution of a nonautonomous difference equation.

Given a NDS (X, f_∞), it induces a NDS $(\mathcal{K}(X), \overline{f_\infty})$, where $\mathcal{K}(X)$ is the hyperspace of all non-empty compact subsets of X endowed with the Vietoris topology and $\overline{f_n} : \mathcal{K}(X) \to \mathcal{K}(X)$ is the continuous function induced by f_n. Here, $\overline{f_n}(A) = f_n(A)$ for each $A \in \mathcal{K}(X)$. Thus, $\overline{f}_1^n = \overline{f_n} \circ \cdots \circ \overline{f_2} \circ \overline{f_1}$.

Usually, an autonomous discrete dynamical system can be regarded as describing dynamics of individuals (points) in the state space X and its induced continuous function on the hyperspace as a form of collective behavior. This interpretation raises a natural question: does individual chaos imply collective chaos? and conversely? For autonomous systems it is known that $(\mathcal{K}(X), \overline{f_\infty})$ is weakly mixing if and only if (X, f) is weakly mixing, and the latter condition is equivalent to the transitivity of $(\mathcal{K}(X), \overline{f_\infty})$ (see [19, Theorem 2.1]). In [14], A. Khan and P. Kumar studied chaotic properties in the sense of Devaney for NDS, by considering the hyper-

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space $\mathcal{K}(X)$ with the w^e-topology. Some results concerning chaotic properties in NDS were obtained in [2,7,12,17,18,24].

In Section 3 we study dynamical properties related to transitivity in a NDS (X, f_∞) and the induced NDS $(\mathcal{K}(X), \overline{f_\infty})$. In particular, we give some examples to show that [19, Theorem 2.1] cannot be extend to NDS and, however, that some implications remain true. We proved that if $(\mathcal{K}(X), \overline{f_\infty})$ is weakly mixing, then (X, f_∞) is weakly mixing (Corollary 3.9), and that the transitivity of $(\mathcal{K}(X), \overline{f_\infty})$ implies that (X, f_∞) satisfies Banks's condition and is transitive (Propositions 3.5–3.6). The property which is preserved in both directions is to be weakly mixing of all orders (Proposition 3.10).

In Section 4 we study several properties related to chaos in the sense of Devaney. Given a metric space X and a continuous function f: $X \to X$, an autonomous discrete system (X, f) is said to be chaotic in the sense of Devaney ([6]) if it is transitive, has dense set of periodic points and is sensitive (dependence on initial conditions). It is a well-known fact that transitivity and density of periodic points imply the sensitivity of f (see [4] and [11, Theorem 9.20]) so that it is natural to ask if this result remains true for NDS (some conditions under this result is also valid for NDS are given in [27]). We address this question for NDS on the interval. Example 4.4 shows that the answer is negative for this relevant class of NDS. We also present an example that proves that the classic result that transitivity implies chaos in the sense of Devaney for autonomous dynamical systems on the interval ([1,21,25]) fails to be true for NDS.

As a connection between Section 3 and Section 4, we can say that for an autonomous discrete system (X, f), there is no relation between Devaney chaos for (X, f) and its associated hyperspace $(\mathcal{K}(X), \overline{f})$ (see [10]):

(X, f) is Devaney chaotic $\not\Rightarrow$ $(\mathcal{K}(X), \overline{f})$ is Devaney chaotic
$(\mathcal{K}(X), \overline{f})$ is Devaney chaotic $\not\Rightarrow$ (X, f) is Devaney chaotic

2. Preliminaries

Given a subset A of a topological space X, $\mathrm{Cl}_X(A)$ and $\mathrm{Int}_X(A)$ denote the closure and the interior of A in X, respectively.

A NDS (X, f_∞) is *point transitive* if there exists $x \in X$ with dense orbit in X, i.e. $\mathrm{Cl}_X(\mathrm{orb}(x, f_\infty)) = X$. In this case, we say that x is a *transitive point* of (X, f_∞). Also, (X, f_∞) is *topologically transitive* if for any two non-empty open sets U and V in X, there exists a positive integer k such that $f_1^k(U) \cap V \neq \emptyset$. A NDS (X, f_∞) is said to satisfy *Banks's condition* if for any three non-empty open sets U, V, W in X, there exists a positive integer k such that $f_1^k(U) \cap V \neq \emptyset$ and $f_1^k(U) \cap W \neq \emptyset$. We say that (X, f_∞) is *weakly mixing* if for any four non-empty open sets U_1, U_2, V_1, V_2 in X, there exists a positive integer k such that $f_1^k(U_i) \cap V_i \neq \emptyset$, for each $i \in \{1, 2\}$. It is clear that if (X, f_∞) is weakly mixing, then it has Banks's condition and, this implies that it is transitive.

For a NDS (X, f_∞) we put $X^2 = X \times X$ and $(f_\infty)^2 = (g_1, g_2, \ldots, g_n, \ldots)$, where $g_n = f_n \times f_n$ for each positive integer n. Thus, $(X^2, (f_\infty)^2)$ is a NDS. Note that

$$\begin{aligned} g_1^n &= g_n \circ g_{n-1} \circ \cdots \circ g_2 \circ g_1 \\ &= (f_n \times f_n) \circ (f_{n-1} \times f_{n-1}) \circ \cdots \circ (f_2 \times f_2) \circ (f_1 \times f_1) = f_1^n \times f_1^n. \end{aligned}$$

In general, for a positive integer m, we define the nonautonomous discrete dynamical system $(X^m, (f_\infty)^m)$, where

$$X^m = \underbrace{X \times \cdots \times X}_{m-times}$$

and $(f_\infty)^m = (g_1, \ldots, g_n, \ldots)$, where

$$g_n = \underbrace{f_n \times \cdots \times f_n}_{m-times},$$

for each positive integer n.

We say that (X, f_∞) is *weakly mixing of order* m ($m \geq 2$) if $(X^m, (f_\infty)^m)$ is transitive, i.e. for any non-empty open sets $U_1, U_2, \ldots, U_m, V_1, V_2, \ldots, V_m$ there is an integer $n > 0$ such that $f_1^n(U_i) \cap V_i \neq \emptyset$ for each $1 \leq i \leq m$.

Let X be a topological space. The symbol $\mathcal{K}(X)$ will denote the hyperspace of all non-empty compact subsets of X endowed with the Vietoris topology. Let us recall that the following sets constitute a base of open sets for Vietoris topology:

$$\begin{aligned} \langle U_1, \ldots, U_k \rangle := \{K \in \mathcal{K}(X) : K \subset \bigcup_{i=1}^{k} U_i \text{ and} \\ K \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, k\}\}, \end{aligned}$$

where $U_1, \ldots, U_k$ are non-empty open subsets of X.

Given a metric space (X, d), a point $x \in X$ and $A \in \mathcal{K}(X)$, let $d(x, A) = \inf\{d(x, a) : a \in A\}$. For every $\epsilon > 0$, we define the *open d-ball in X about A and radius* ϵ by $N_d(\epsilon, A) = \{x \in X : d(x, A) < \epsilon\} = \bigcup_{a \in A} B(\epsilon, a)$, where $B(\epsilon, a)$ denotes the open ball in X centred at a and radius ϵ. We define in $\mathcal{K}(X)$ the *Hausdorff metric induced by d*, denoted by H_d, as follows

$$H_d(A, B) = \inf\{\epsilon > 0 : A \subset N_d(\epsilon, B) \text{ and } B \subset N_d(\epsilon, A)\},$$

where $A, B \in \mathcal{K}(X)$. In [13, Theorem 2.2] it is proved that, indeed, H_d is a metric on $\mathcal{K}(X)$. Moreover, it is known [13, Theorem 3.1] that the topology induced by the Hausdorff metric coincides with the Vietoris topology. We denote by $N(\epsilon, A)$ (respectively, by H) the generalized open d-ball $N_d(\epsilon, A)$ (respectively, the metric H_d) when it is clear for the metric d that is used.

If X is a compact metric space and $A, B \in \mathcal{K}(X)$, it follows that $H(A, B) < \epsilon$ if and only if $A \subset N(\epsilon, B)$ and $B \subset N(\epsilon, A)$.

Given a continuous function f: $X \to X$, it induces a continuous function on $\mathcal{K}(X)$, $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by $\overline{f}(K) = f(K)$ for every $K \in \mathcal{K}(X)$. It is known that the continuity of f implies the continuity of $\overline{f}$ (see [13, Lemma 13.3]).

Let (X, f_∞) be a NDS and $\overline{f_n}$ the induced continuous function of f_n on $\mathcal{K}(X)$, for each positive integer n. Then, the sequence $\overline{f_\infty} = (\overline{f_1}, \overline{f_2}, \ldots, \overline{f_n}, \ldots)$ induces a nonautonomous discrete dynamical system $(\mathcal{K}(X), \overline{f_\infty})$. In this case, $\overline{f}_1^n = \overline{f_n} \circ \cdots \circ \overline{f_2} \circ \overline{f_1}$. Note that $\overline{f}_1^n = \overline{f_1^n}$. As usual, $\mathbf{I}$ denotes the unit interval and $\mathbb{R}$ the real numbers equipped with the usual topology.

3. Transitivity and related properties

In the following theorem, the equivalence of (1) and (3) was independently showed by Banks [3] and Peris [19]. Moreover, in [5] was already shown that (2) implies (1).

The results in this section are motivated by the following theorem of autonomous discrete dynamical systems. Our aim is to study which implications remain valid for the case of a NDS. We construct some examples to show that some implications can not be extended to nonautonomous discrete dynamical systems.

Theorem 3.1. *[19, Theorem 2.1] Let f: $X \to X$ be a continuous function on a topological space X. Then the following conditions are equivalent:*

(1) *(X, f) is weakly mixing.*
(2) *$(\mathcal{K}(X), \overline{f})$ is weakly mixing.*
(3) *$(\mathcal{K}(X), \overline{f})$ is transitive.*

Clearly, (2) implies (3) even in nonautonomous discrete dynamical systems. The next example shows that (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (2) are false for a NDS. Given two points $(a, b), (c, d) \in \mathbb{R}^2$, $[(a, b), (c, d)]$ stands for the segment whose endpoints are (a, b) and (c, d), respectively.

Example 3.2. There is a NDS $(\mathbf{I}, f_\infty)$ which is weakly mixing, but $(\mathcal{K}(\mathbf{I}), \overline{f_\infty})$ is not transitive.