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Traveling wave solutions for the hyperbolic Cahn–Allen equation

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ABSTRACT

Traveling wave solutions of the hyperbolic Cahn–Allen equation are obtained using the first integral method, which follows from well-known Hilbert–Nullstellensatz theorem. The obtained complete class of traveling waves consists of continual and singular solutions. Continual solutions are represented by tanh-profiles and singular solutions exhibit unbounded discontinuity at the origin of coordinate system. With the neglecting inertia of the dynamical system, the obtained traveling waves include the previous solutions for the parabolic Cahn–Allen equation.

1. Introduction

The Cahn–Allen partial differential equation (CA-PDE) was suggested for the anti-phase boundary motion [1,2] and used in a wide spectrum of applications (see Ref. [3] and references therein). Being a useful tool within the phase-field models [4], the CA-PDE provides a framework for the mathematical description of free-boundary problems. One of important analytical solutions is related to traveling wave solutions (see Refs. [5,6] and references therein). Nowadays, one of convenient and complete ways in obtaining traveling waves lies in the use of the first integral method [7]. This method was introduced for a reliable treatment of the non-linear PDEs and over the last decades it has an intense period of its applicability [8,9]. This method can be considered as one of particular cases of *the direct method* [10] which generalizes the use of equivalent methods in finding exact solutions of PDE reduced to ODE [11]. Indeed, to date, several useful methods for obtaining solitons and traveling waves were developed. Among them, for instance, exist *the tanh method* [12], *G'/G–expansion method* [13], and the other powerful method formulated as *the rank analytical technique* [14] applicable to a wide spectrum of nonlinear evolution PDEs [15]. To investigate some kinds of specific traveling waves, for example, solitary or periodic cusp waves, *a phase–plane analysis* seems to have a strong operability [16]. In the present work, we use the first integral method due to its evidence and simple applicability to parabolic and hyperbolic types of PDEs with the obtaining of complete set of traveling wave solutions [17,18].

Special attention is given to the hyperbolic equations [19–23], especially, to CA-PDE with regard to its application in the field of fast phase transitions [23,24]. For the hyperbolic CA-PDE, a traveling wave solution in a particular form of tanh-function has been assumed [25,26]. So far, a general set of exactly obtained traveling waves for the hyperbolic CA-PDE is absent. Therefore, the main purpose of the present work is to find a general set of traveling waves in an exact analytical solution of the hyperbolic CA-PDE.

The hyperbolic equation for the order parameter ϕ is given by [24]

$$\tau_R \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial t} = D\nabla^2 \phi - M_\phi \frac{df(\phi)}{d\phi}, \qquad (1)$$

where t is the time, D is the diffusion parameter for the order parameter ϕ, M_ϕ is the mobility of ϕ and τ_R is the relaxation time for the gradient flow $\partial\phi/\partial t$. In the damped-wave Eq. (1) the inertial term $\tau_R \partial^2\phi/\partial t^2$ changes the type of the equation from usual dissipative parabolic type to the hyperbolic one with the drastic exchange of its analytical properties (as it has been shown for particular case of CA-PDE [27]). Physically reasonable applications of this equation are in processes with rapidly moving interfaces, in comparison with data of atomistic simulations on crystal growth kinetics, fast motion by mean curvature under large driving force of transformation between non-equilibrium states, inertial dynamics with dissipation, etc. (see the work [26] and references therein).

With the free energy density $f(\phi) = \frac{1}{4}(\phi^2 - 1)^2$, Eq. (1) gives

$$\tau_R \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial t} = D\nabla^2 \phi + M_\phi (\phi - \phi^3), \qquad (2)$$

which is the hyperbolic CA-PDE. Introducing the dimesionless relaxation time $\tau = \tau_R M_\phi$, dimensionless coordinate $x^* = x\sqrt{M_\phi/D}$,

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dimensionless time $t^* = M_\phi t$, and the new coordinate $\xi = x^* - ct^*$ which moves with the constant velocity, $c = const$, with the origin at $\phi = 1/2$, Eq. (2) transforms to the one-dimensional ordinary differential equation (ODE)

$$(1-\tau c^2)\frac{d^2\phi}{d\xi^2} + c\frac{d\phi}{d\xi} + \phi - \phi^3 = 0. \quad (3)$$

We shall use the first integral method [7] to check the existence of tanh-functions in the traveling waves of Eq. (3).

2. The first integral method and traveling-wave solutions

Eq. (3) has the trivial constant solutions: +1, −1, and 0. For its non-constant solutions, Eq. (3) is considered as non-linear ODE of the type of traveling wave solution $Q(\phi, \phi', \phi'', \dots) = 0$, where the prime denotes the derivative with respect to ξ. The solution of this ODE can be written in the form of $\phi_i(x^*, t^*) = F(\xi)$, where $i = 1, \dots, m$. Now we introduce a new independent variable and its derivative as $X(\xi) = F(\xi)$ and $Y(\xi) = X'(\xi)$, respectively. According to the first integral method [7], $X(\xi)$ and $Y(\xi)$ are non-trivial solutions of Eq. (3):

$$\frac{dX(\xi)}{d\xi} = Y(\xi), \quad (4)$$

$$(1-\tau c^2)\frac{dY(\xi)}{d\xi} = X^3(\xi) - X(\xi) - cY(\xi), \quad (5)$$

expressed by $X(\xi)$ and $Y(\xi)$ using the following polynomial

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0. \quad (6)$$

The polynomial (6) is known to be the first integral to Eqs. (4) and (5) due to the division theorem[1], if we suppose $a_i(X)$ to be polynomials of X and $a_m(X) \neq 0$. This first integral reduces Eq. (3) to a first order integrable ODE, which must have the exact analytical solutions.

In Eq. (6), we consider X(ξ) and Y(ξ) as independent functions in the complex domain $C(X, Y)$, therefore, $dY/dX = 0$. Following the division theorem [7,28], there exists the polynomial $g(X) + h(X)Y$. Then, we shall write

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{dX}{d\xi} + \frac{\partial q}{\partial Y}\frac{dY}{d\xi} = \frac{\partial q}{\partial X}Y + \frac{\partial q}{\partial Y}\frac{X^3 - X - cY}{1-\tau c^2}$$
$$= [g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i,$$

in the complex domain $C(X, Y)$. From the latest expressions one obtains

$$\frac{\partial q}{\partial X} = \sum_{i=0}^{m}\frac{da_i}{dX}Y^i + \sum_{i=0}^{m} i a_i Y^{i-1}\frac{\partial Y}{\partial X} = \sum_{i=0}^{m}\frac{da_i}{dX}Y^i, \quad (7)$$

where the coefficients a_i for the solution (6) are

$$\sum_{i=0}^{2}\frac{da_i}{dX}Y^{i+1} + \sum_{i=0}^{2} i a_i Y^{i-1}\frac{X^3 - X - cY}{1-\tau c^2}$$
$$= g(X)\sum_{i=0}^{2} a_i(X)Y^i + h(X)Y\sum_{i=0}^{2} a_i(X)Y^i. \quad (8)$$

A number of terms in Eqs. (7) and (8) are chosen as $m = 2$ to reach the required polynomial degree with the correct number of terms in determining the resulting number of unknown coefficients. Then, using the detailed form of Eq. (8) and having equating the coefficients at $Y^i (i = 0, 1, 2, 3)$, one gets

$$Y^3: \quad \dot{a}_2(X) = h(X)a_2(X), \quad (9)$$

$$Y^2: \quad \dot{a}_1(X) = 2a_2(X)\frac{c}{1-\tau c^2} + g(X)a_2(X) + h(X)a_1(X), \quad (10)$$

$$Y^1: \quad \dot{a}_0(X) = 2a_2(X)\frac{X - X^3}{1-\tau c^2} + a_1(X)\frac{c}{1-\tau c^2}$$
$$+ g(X)a_1(X) + h(X)a_0(X), \quad (11)$$

$$Y^0: \quad a_1(X)\frac{X^3 - X}{1-\tau c^2} = g(X)\ a_0(X), \quad (12)$$

where the point means the derivative d/dX. Since $a_i(X)$ are polynomials, from Eq. (9) it follows that $a_2(X) \equiv const$ and $h(X) = 0$. Then, accepting the value $a_2(X) = 1$, Eqs. (9)–(12) are

$$a_2(X) = 1, \quad (13)$$

$$\dot{a}_1(X) = 2\frac{c}{1-\tau c^2} + g(X), \quad (14)$$

$$\dot{a}_0(X) = a_1(X)\frac{c}{1-\tau c^2} - 2\frac{X^3 - X}{1-\tau c^2} + g(X)a_1(X), \quad (15)$$

$$a_1(X)\frac{X^3 - X}{1-\tau c^2} = g(X)a_0(X). \quad (16)$$

Thus, we have found the expressions for the coefficients $a_i(X)$ from Eq. (6).

In searching for solutions of Eq. (3), we assume the condition $1 - \tau c^2 > 0$ which physically means that the interface velocity c cannot overcome and be larger than the maximum speed of disturbance propagation in the field of order parameter ϕ [24,25]. Such condition shrinks the set of all possible solutions (real and imaginary) to the class of real solutions. Solutions of Eq. (3) will be found for two possible cases: **deg**$[g(X)] = 0$ and **deg**$[g(X)] = 1$ which directly follow from Eqs. (8)–(12), **see** Appendix A.

2.1. Case 0

Minding **deg**$[g(X)] = 0$ for Eqs. (13)–(16), one obtains $g(X) = A_1$ and $a_1(X) = 2cX/(1-\tau c^2) + A_1X + A_0$. With these expressions, the integral of Eq. (15) is

$$a_0(X) = -\frac{X^4}{2(1-\tau c^2)} + \frac{X^2}{1-\tau c^2}\left(1 + \frac{c^2}{1-\tau c^2}\right.$$
$$\left. + \frac{3}{2}A_1c + \frac{1-\tau c^2}{2}A_1^2\right) + X\left(\frac{A_0c}{1-\tau c^2} + A_0A_1\right) + d, \quad (17)$$

where d is the constant of integration.

Substituting $a_0(X)$ from Eq. (17) into Eq. (8), multiplying the result by $(1-\tau c^2)^2$ and having equating to zero the coefficients for different degrees of X, one gets

$$X^4: \quad 2c + \frac{3}{2}A_1\left(1-\tau c^2\right) = 0, \quad (18)$$

$$X^3: \quad A_0\left(1-\tau c^2\right) = 0, \quad (19)$$

$$X^2: \quad 2c + 2A_1\left(1-\tau c^2\right) + A_1c^2 + \frac{3}{2}A_1^2c\left(1-\tau c^2\right)$$
$$+ \frac{1}{2}\left(1-\tau c^2\right)^2 A_1^3 = 0, \quad (20)$$

[1] The division theorem" has been formulated in Ref. [7]. as a particular case of the Hilbert–Nullstellensatz theorem about characterization of maximal ideals in polynomial rings [28].