Chaos, Solitons and Fractals 94 (2017) 112–118

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Review

Periodic solutions of some classes of continuous third-order differential equations

Makhlouf Amar*, Debbabi Djamel

Department of Mathematics, Laboratory LMA, University Badji Mokhtar, Elhadjar, Annaba 23, Algeria

A R T I C L E I N F O

Article history:
Received 28 August 2016
Revised 25 November 2016
Accepted 28 November 2016

MSC:
Primary: 37G15
37C80
37C30

Keywords:
Periodic solution
Third order differential equations
Averaging theory

A B S T R A C T

We study the periodic solutions of the third-order differential equations of the form $\dddot{x} \pm x^n = \mu f(t)$, or $\dddot{x} \pm |x|^n = \mu f(t)$, where $n = 2, 3, \ldots, f(t)$ is a continuous $T-$ periodic function such that $\int_0^T f(t)dt \neq 0$, and μ is a positive small parameter. Note that the differential equations $\dddot{x} \pm x^n = \mu f(t)$ are only continuous in t and smooth in x, and that the differential equations $\dddot{x} \pm |x|^n = \mu f(t)$ are only continuous in t and locally-Lipschitz in x. We also study the stability of the periodic solutions.

1. Introduction

The periodic solutions of the second-order differential equations

$$\ddot{x} + x^3 = f(t),$$

where $f(t)$ is a T-periodic function have been studied by several authors. Mainly these authors study when an equilibrium or a periodic orbit of an autonomous differential system, as the system $\ddot{x} + x^3 = 0$, can be continued as a periodic solution when the autonomous system is periodically perturbed, see for instance the papers [5,6,9,10]. The main tool used by these authors for obtaining their results is the Brouwer degree theory.

The objective of this paper is to extend the mentioned results in two new directions:

• First instead of working with second-order differential equations we shall work with third-order differential equations.

• Second instead of working with the particular autonomous system $\dddot{x} + x^3 = 0$ we shall work with the autonomous systems $\dddot{x} \pm x^n = 0$ or $\dddot{x} \pm |x|^n = 0$ for all

$n \geq 2$.

If $f(t)$ is a T-periodic function, then we shall study the periodic orbits and their kind of stability of the non-autonomous third-order differential equations

$$\dddot{x} \pm x^n = \mu f(t), \tag{1}$$

which are only continuous in t and smooth in x, and also we shall study the periodic orbits and their kind of stability of the differential equations

$$\dddot{x} \pm |x|^n = \mu f(t), \tag{2}$$

which are only continuous in t and locally-Lipschitz in x.

We remark that almost there are no previous works studying periodic orbits of differential systems which are only locally-Lipschitz, see for instance [7].

2. Statement of the main results

Our main results are the following four theorems.

Theorem 1. *Consider the third-order differential equation*

$$\dddot{x} + x^n = \mu f(t), \tag{3}$$

where $n = 2, 3, \ldots, f(t)$ is continuous, T periodic function such that, $\int_0^T f(t)dt \neq 0$ and $\mu > 0$ is a small parameter.

For n even, $\int_0^T f(t)dt > 0$ and $\mu > 0$ suffciently small there exist two unstable periodic solutions $x_+(t, \mu)$ and $x_-(t, \mu)$ of period T of the differential Eq. (3) such that

$$x_+(0, \mu) = \mu^{\frac{1}{n}} \left(\frac{1}{T} \int_0^T f(t)dt \right)^{\frac{1}{n}} + O(\mu^{\frac{(n+2)}{3n}})$$

* Corresponding author.
E-mail addresses: makhloufamar@yahoo.fr (M. Amar), djamel.debbabi@gmail.com (D. Djamel).

http://dx.doi.org/10.1016/j.chaos.2016.11.012

and

$$x_-(0,\mu) = -\mu^{\frac{1}{n}}\left(\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}} + O(\mu^{\frac{(n+2)}{3n}}).$$

For n odd, there exists only one unstable periodic solution x(t, μ) of period T of the differential Eq. (3) such that

$$x(0,\mu) = \mu^{\frac{1}{n}}\left(\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}} + O(\mu^{\frac{(n+2)}{3n}}).$$

Theorem 1 is proved in Section 2.

Theorem 2. *Consider the third-order differential equation*

$$\dddot{x} - x^n = \mu f(t), \tag{4}$$

where $n = 2, 3, \ldots, f(t)$ is continuous, T periodic function such that, $\int_0^T f(t)dt \neq 0$ and $\mu > 0$ is a small parameter.

For n even, $\int_0^T f(t)dt < 0$ and $\mu > 0$ suffciently small there exist two unstable periodic solutions $x_+(t,\mu)$ and $x_-(t,\mu)$ of period T of the differential Eq. (4) such that

$$x_+(0,\mu) = \mu^{\frac{1}{n}}\left(-\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}} + O(\mu^{\frac{(n+2)}{3n}})$$

and

$$x_-(0,\mu) = -\mu^{\frac{1}{n}}\left(-\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}} + O(\mu^{\frac{(n+2)}{3n}}).$$

For n odd, there exists only one unstable periodic solution x(t, μ) of period T of the differential Eq. (4) such that

$$x(0,\mu) = \mu^{\frac{1}{n}}\left(-\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}} + O(\mu^{\frac{(n+2)}{3n}}).$$

Theorem 3. *Consider the third-order differential equation*

$$\dddot{x} + |x|^n = \mu f(t), \tag{5}$$

where $n = 2, 3, \ldots, f(t)$ is continuous, T periodic function such that, $\int_0^T f(t)dt \neq 0$ and $\mu > 0$ is a small parameter. For $n \geq 2$, $\int_0^T f(t)dt > 0$ and $\mu > 0$ suffciently small there exist two unstable periodic solutions $x_+(t,\mu)$ and $x_-(t,\mu)$ of period T of the differential Eq. (5) such that

$$x_+(0,\mu) = \mu^{\frac{1}{n}}\left(\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}} + O(\mu^{\frac{(n+2)}{3n}})$$

and

$$x_-(0,\mu) = -\mu^{\frac{1}{n}}\left(\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}} + O(\mu^{\frac{(n+2)}{3n}}).$$

Theorem 3 is proved in Section 2.

Theorem 4. *Consider the third-order differential equation*

$$\dddot{x} - |x|^n = \mu f(t), \tag{6}$$

where $n = 2, 3, \ldots, f(t)$ is continuous, T periodic function such that, $\int_0^T f(t)dt \neq 0$ and $\mu > 0$ is a small parameter. For $n \geq 2$, $\int_0^T f(t)dt < 0$ and $\mu > 0$ suffciently small there exist two unstable periodic solutions $x_+(t,\mu)$ and $x_-(t,\mu)$ of period T of the differential Eq. (6) such that

$$x_+(0,\mu) = \mu^{\frac{1}{n}}\left(-\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}} + O(\mu^{\frac{(n+2)}{3n}})$$

and

$$x_-(0,\mu) = -\mu^{\frac{1}{n}}\left(-\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}} + O(\mu^{\frac{(n+2)}{3n}}).$$

The following two corollaries follow easily from the previous four theorems.

Corollary 1. *(a) For $\mu > 0$ sufficiently small the equation $\dddot{x} + x^2 = \mu(\sin^3 t + 1)$ has two unstable periodic solutions $x_+(t,\mu)$ and $x_-(t,\mu)$ such that*

$$x_+(0,\mu) = \sqrt{\mu} + O(\mu^{\frac{2}{3}}) \text{ and } x_-(0,\mu) = -\sqrt{\mu} + O(\mu^{\frac{2}{3}}).$$

(b) For $\mu > 0$ sufficiently small the equation $\dddot{x} - x^3 = \mu\cos^2 t$ has only one unstable periodic solution x(t, μ) such that

$$x(0,\mu) = \sqrt[3]{-\mu/2} + O(\mu^{\frac{5}{9}}).$$

Corollary 2. *(c) For $\mu > 0$ sufficiently small the equation $\dddot{x} + |x|^6 = \mu\cos^2 t$ has two unstable periodic solutions $x_+(t,\mu)$ and $x_-(t,\mu)$ such that*

$$x_+(0,\mu) = \sqrt[6]{\mu/2} + O(\mu^{\frac{4}{9}}) \text{ and } x_-(0,\mu) = -\sqrt[6]{\mu/2} + O(\mu^{\frac{4}{9}}).$$

(d) For $\mu > 0$ sufficiently small the equation $\dddot{x} - |x|^5 = -\mu\sin^2 t$ has two unstable periodic solutions $x_+(t,\mu)$ and $x_-(t,\mu)$ such that

$$x_+(0,\mu) = \sqrt[5]{\mu/2} + O(\mu^{\frac{7}{15}}) \text{ and } x_-(0,\mu) = -\sqrt[5]{\mu/2} + O(\mu^{\frac{7}{15}}).$$

3. Proof of the results

In this section we shall prove Theorems 1–4 and Corollaries 1 and 2.

Proof of Theorem 1. Under the assumptions of Theorem 1 we write the third-order differential equation as the differential system of first order

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= -x^n + \mu f(t).\end{aligned} \tag{7}$$

Doing the change of variables

$$\begin{aligned}&x = \varepsilon^{3/(n-1)}X, \quad y = \varepsilon^{(n+2)/(n-1)}Y,\\ &z = \varepsilon^{(2n+1)/(n-1)}Z, \quad \mu = \varepsilon^{3n/(n-1)},\end{aligned} \tag{8}$$

with $\varepsilon > 0$, the differential system (7) becomes

$$\begin{aligned}\dot{X} &= \varepsilon Y,\\ \dot{Y} &= \varepsilon Z,\\ \dot{Z} &= \varepsilon(-X^n + f(t)).\end{aligned} \tag{9}$$

We note that the change of variables (8) is well defined because $n > 1$. Now we apply the averaging theory of first order of the appendix. Using the notation of Theorem 5 of the appendix system (9) can be written as system (18) with $x = (X, Y, Z)$, $H = (Y, Z, -X^n + f(t))$, and $R = (0, 0, 0)$. The averaged function $h(z)$ given in (19) for system (9) becomes

$$h(X,Y,Z) = \left(Y, Z, -X^n + \frac{1}{T}\int_0^T f(t)dt\right). \tag{10}$$

If n is even and $\int_0^T f(t)dt > 0$ then the function $h(X, Y, Z)$ has two unique zeros

$$(X_+^*, Y_+^*, Z_+^*) = \left(\left(\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}}, 0, 0\right)$$

and

$$(X_-^*, Y_-^*, Z_-^*) = \left(-\left(\frac{1}{T}\int_0^T f(t)dt\right)^{\frac{1}{n}}, 0, 0\right).$$

The Jacobian of the function $h(X, Y, Z)$ at the zero (X_+^*, Y_+^*, Z_+^*) is $-nX_+^{*(n-1)} < 0$. By Theorem 5 and Remark 2 we deduce that