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Equivariant bifurcation in a coupled complex-valued neural network rings☆

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ABSTRACT

Network with interacting loops and time delays are common in physiological systems. In the past few years, the dynamic behaviors of coupled interacting loops neural networks have been widely studied due to their extensive applications in classification of pattern recognition, signal processing, image processing, engineering optimization and animal locomotion, and other areas, see the references therein. In a large amount of applications, complex signals often occur and the complex-valued recurrent neural networks are preferable. In this paper, we study a complex value Hopfield-type network that consists of a pair of one-way rings each with four neurons and two-way coupling between each ring. We discuss the spatio-temporal patterns of bifurcating periodic oscillations by using the symmetric bifurcation theory of delay differential equations combined with representation theory of Lie groups. The existence of multiple branches of bifurcating periodic solution is obtained. We also found that the spatio-temporal patterns of bifurcating periodic oscillations alternate according to the change of the propagation time delay in the coupling, i.e., different ranges of delays correspond to different patterns of neural network oscillators. The oscillations of corresponding neurons in the two loops can be in phase or anti-phase depending on the parameters and delay. Some numerical simulations support our analysis results.

1. Introduction

Neural network which is composed of complex-valued neurons is extensively studied in the various fields [1–3]. Generally, complex-valued neural networks have different and more complicated properties than real-valued neural networks. Usually, complex-valued neural networks make it possible to solve some problems which cannot be solved with their real-valued counterparts. For example, the XOR problem and the detection of symmetry problem cannot be solved with a single real-valued neuron, but they can be solved with a single complex-valued neuron with the orthogonal decision boundaries, which reveals the potent computational power of complex-valued neurons [4]. Thus, it is important to investigate the dynamical behaviors of complex-valued recurrent neural networks. Recently, many properties of complex valued neural network models have been studied, such as exponential stability of the equilibrium point, Hopf bifurcation, global asymptotic stability, boundedness and complete stability and so on. See [5–9].

Time delays have been incorporated into neural network models by many authors, since it occurs in signal transmissions in the electronic implementation of neural networks, which may influence the dynamical behaviors such as oscillation, bifurcation and instability. From the mathematical point of view, the presence of delays makes the problem harder to handle. In fact, the state vector characterizing a nonlinear delayed system evolves in an infinite dimensional functional space [10,11].

The theory of spatio-temporal pattern formation in systems of coupled non-linear oscillators with symmetry has grown extensively in recent years. There are many phenomena which can be modeled as symmetric systems of non-linear coupled oscillators and are examples of symmetry-breaking bifurcation. These systems often can described by a network of units called cells (oscillators) and can display many different kinds of dynamics [12,13]. For the symmetrical dynamical system described by ordinary differential equations, Golubisky et al. provides the symmetrical (equivariant) bifurcation theory.

A coupled oscillator network has discrete spatial structure but continuous temporal structure be modeled as a structured system with properties such as geometry and symmetry. Its impact has been felt in a wide variety of fields of applied science. An

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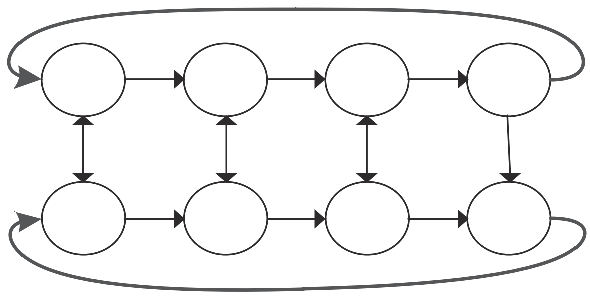

Fig. 1. The architecture of the model (1.1).

important related example in biology is "animal locomotion", see [14–18].

In this paper, a complex value delayed neuron Hopfield neural networks with a ring topology consists of two coupling unidirectional rings, each with four oscillators are given. In this model, the states, connection weights and activation functions of the complex-valued neural networks are all complex-valued. The model has symmetric group $\Gamma = Z_4 \times Z_2$, that means the global symmetry Z_2 and internal symmetry Z_4. See Fig. 1.

The case leads to the following system of delay differential equations:

$$\begin{cases} \dot{z}_1(t) = (a+ib)z_1(t) + (c+id)f(z_2(t)) + (m+in)f(w_1(t-\tau)), \\ \dot{z}_2(t) = (a+ib)z_2(t) + (c+id)f(z_3(t)) + (m+in)f(w_2(t-\tau)), \\ \dot{z}_3(t) = (a+ib)z_3(t) + (c+id)f(z_4(t)) + (m+in)f(w_3(t-\tau)), \\ \dot{z}_4(t) = (a+ib)z_4(t) + (c+id)f(z_1(t)) + (m+in)f(w_4(t-\tau)), \\ \dot{w}_1(t) = (a+ib)w_1(t) + (c+id)f(w_2(t)) + (m+in)f(z_1(t-\tau)), \\ \dot{w}_2(t) = (a+ib)w_2(t) + (c+id)f(w_3(t)) + (m+in)f(z_2(t-\tau)), \\ \dot{w}_3(t) = (a+ib)w_3(t) + (c+id)f(w_4(t)) + (m+in)f(z_3(t-\tau)), \\ \dot{w}_4(t) = (a+ib)w_4(t) + (c+id)f(w_1(t)) + (m+in)f(z_4(t-\tau)), \end{cases} \tag{1.1}$$

where $\tau \geq 0$ is the time delay and the coefficients a, b, c, d, m, n are all real values. Activation function $f(z) = (\tanh(x) + \tanh(y)) + i(\tanh(x) + \tanh(y))$ $(z = x + iy)$is common to all neurons.

The general theory of Hopf bifurcation applied to systems of symmetrically coupled identical oscillators was developed by many mathematicians, chief among whom were M. Golubitsky and I. Stewart [19]. Recently, the works of addressed the effects of both internal and global symmetries are analyzed to determine exactly what kind of solutions are possible [20,21]. In the next section we focus on the linear stability analysis of the trivial equilibrium. This then leads us to a discussion of the bifurcations of the trivial equilibrium. We consider the coupled system's dynamics near a multiple Hopf bifurcation. We show that the structure of system (1.1) can be represented by a group $Z_4 \times Z_2$. There is a fully symmetric solution that loses stability as a parameter varies, and this loss of stability is due to the crossing of imaginary eigenvalues through the imaginary axis and the Hopf bifurcation to periodic solutions appears. In Section 3, we also obtain some important results about the spontaneous bifurcations of multiple branches of periodic solutions: though the system (1.1) has the global symmetry Z_2 and internal symmetry Z_4, there are several kinds spatio-temporal patterns which can characterize the coordination between the neurons in two rings during cyclic movements. Numerical simulations is given in Section 4. In Section 5, we also point out that the theoretical results obtained can be applied to practical systems with $Z_2 \times Z_4$ symmetry, such as CPG model of a quadrupedal locomotor.

2. Elementary analysis

In order to investigate the dynamic behavior of the complex-valued neurons, it is more convenient to work with the equations in a real form. This can be done by introducing

$z_j(t) = x_j(t) + iy_j(t)$ $w_j(t) = u_j(t) + iv_j(t)$ be a solution of (1.1). By taking the real and imaginary parts, we have

$$\begin{cases} \dot{x}_j(t) = ax_j(t) - by_j(t) + cx_{j+1}(t) - dy_{j+1}(t) \\ \qquad + mu_j(t-\tau) - nv_j(t-\tau) \\ \qquad + \frac{c}{3!}[(x_{j+1}^3(t) - 3x_{j+1}^2(t)y_{j+1}(t))] \\ \qquad + \frac{d}{3!}[(y_{j+1}^3(t) - 3x_{j+1}(t)y_{j+1}^2(t))] \\ \qquad + \frac{m}{3!}[(u_j^3(t-\tau) - 3u_j^2(t-\tau)v_j(t-\tau))] \\ \qquad + \frac{n}{3!}[(v_j^3(t-\tau) - 3u_j(t-\tau)v_j^2(t-\tau))], \\ \dot{y}_j(t) = bx_j(t) + ay_j(t) + dx_{j+1}(t) + cy_{j+1}(t) \\ \qquad + nu_j(t-\tau) + mv_j(t-\tau) \\ \qquad + \frac{d}{3!}[(x_{j+1}^3(t) - 3x_{j+1}^2(t)y_{j+1}(t))] \\ \qquad - \frac{c}{3!}[(y_{j+1}^3(t) - 3x_{j+1}(t)y_{j+1}^2(t))] \\ \qquad + \frac{n}{3!}[(u_j^3(t-\tau) - 3u_j^2(t-\tau)v_j(t-\tau))] \\ \qquad - \frac{m}{3!}[(v_j^3(t-\tau) - 3u_j(t-\tau)v_j^2(t-\tau))], \\ \dot{u}_j(t) = au_j(t) - bv_j(t) + cu_{j+1}(t) - dv_{j+1}(t) \\ \qquad + mx_j(t-\tau) - ny_j(t-\tau) \\ \qquad + \frac{c}{3!}[(u_{j+1}^3(t) - 3u_{j+1}^2(t)v_{j+1}(t))] \\ \qquad + \frac{d}{3!}[(v_{j+1}^3(t) - 3u_{j+1}(t)v_{j+1}^2(t))] \\ \qquad + \frac{m}{3!}[(x_j^3(t-\tau) - 3x_j^2(t-\tau)y_j(t-\tau))] \\ \qquad + \frac{n}{3!}[(y_j^3(t-\tau) - 3x_j(t-\tau)y_j^2(t-\tau))], \\ \dot{v}_j(t) = bu_j(t) + av_j(t) + du_{j+1}(t) + cv_{j+1}(t) \\ \qquad + nx_j(t-\tau) + my_j(t-\tau) \\ \qquad + \frac{d}{3!}[(u_{j+1}^3(t) - 3u_{j+1}^2(t)v_{j+1}(t))] \\ \qquad - \frac{c}{3!}[(v_{j+1}^3(t) - 3u_{j+1}(t)v_{j+1}^2(t))] \\ \qquad + \frac{n}{3!}[(x_j^3(t-\tau) - 3x_j^2(t-\tau)y_j(t-\tau))] \\ \qquad - \frac{m}{3!}[(y_j^3(t-\tau) - 3x_j(t-\tau)y_j^2(t-\tau))], \\ \qquad j = 1, 2, 3, 4. \end{cases} \tag{2.1}$$

This shows that Eq. (1.1) is equivalent to Eq. (2.1). Clearly, Eq. (2.1) is a real differential equation with a delay.

It is clear that $(x_j, y_j, u_j, v_j) = (0, 0, 0, 0)$ $(j = 1, 2, 3, 4)$ is an equilibrium point of Eq. (2.1). The linearization of Eq. (2.1) at the origin leads to

$$\begin{cases} \dot{x}_j(t) = ax_j(t) - by_j(t) + cx_{j+1}(t) - dy_{j+1}(t) \\ \qquad + mu_j(t-\tau) - nv_j(t-\tau), \\ \dot{y}_j(t) = bx_j(t) + ay_j(t) + dx_{j+1}(t) + cy_{j+1}(t) \\ \qquad + nu_j(t-\tau) + mv_j(t-\tau), \\ \dot{u}_j(t) = au_j(t) - bv_j(t) + cu_{j+1}(t) - dv_{j+1}(t) \\ \qquad + mx_j(t-\tau) - ny_j(t-\tau), \\ \dot{v}_j(t) = bu_j(t) + av_j(t) + du_{j+1}(t) + cv_{j+1}(t) \\ \qquad + nx_j(t-\tau) + my_j(t-\tau), \\ \qquad j = 1, 2, 3, 4. \end{cases} \tag{2.2}$$

Let $C([-\tau, 0], \mathrm{R}^{16})$ denote the Banach space of continuous mapping from $[-\tau, 0]$ into R^{16} equipped with the supremum norm $\| \varphi \| = \sup_{-\tau \leq \theta \leq 0} | \varphi(\theta) |$ for $\varphi \in C([-\tau, 0], \mathrm{R}^{16})$. Let $\sigma \in \mathrm{R}$, $A \geq 0$,