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A generalized business cycle model with delays in gross product and capital stock

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ABSTRACT

In this work, we propose a delayed business cycle model with general investment function. The time delays are introduced into gross product and capital stock, respectively. We first prove that the model is mathematically and economically well posed. In addition, the stability of the economic equilibrium and the existence of Hopf bifurcation are investigated. Our main results show that both time delays can cause the macro-economic system to fluctuate and the economic equilibrium to lose or gain its stability. Moreover, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are determined by means of the normal form method and center manifold theory. Furthermore, the models and results presented in many previous studies are improved and generalized.

1. Introduction

In this paper, we consider our delayed business cycle model with general investment function proposed in [1]:

$$\begin{cases} \dfrac{dY}{dt} = \alpha[I(Y(t), K(t)) - \gamma Y(t)], \\ \dfrac{dK}{dt} = I(Y(t-\tau_1), K(t)) - \delta K(t), \end{cases} \tag{1}$$

where $Y(t)$ and $K(t)$ denote the gross product and capital stock at time t, respectively. The parameters α and δ are, respectively, the adjustment coefficient in the goods market and the depreciation rate of the capital stock. Further, the coefficient $\gamma \in (0, 1)$ represents the propensity to save. The investment function is represented by $I(Y, K)$ and it is assumed to be continuously differentiable in $\mathbb{R}^2$ with $\frac{\partial I}{\partial Y} > 0$ and $\frac{\partial I}{\partial K} < 0$. Finally, τ_1 is the time delay between the decision of investment and implementation.

System (1) is a generalization of many delayed business cycle models existing in the literature (see for example, [2–6]) when the investment function $I(Y, K) = I(Y) + qK$, where $q < 0$. In addition, our results presented in [1] extended and improved the corresponding results obtained by Zhang and Wei [4] about the stability analysis and the direction of the Hopf bifurcation.

On the other hand, the investment function in capital accumulation depends on the income and the capital stock both at the past time, and also at the different time delays [7]. For this reason, we consider the following model

$$\begin{cases} \dfrac{dY}{dt} = \alpha[I(Y(t), K(t)) - \gamma Y(t)], \\ \dfrac{dK}{dt} = I(Y(t-\tau_1), K(t-\tau_2)) - \delta K(t). \end{cases} \tag{2}$$

The importance of our new delayed business cycle model (2) is that it includes many special cases. For example, when $\tau_2 = 0$, we get the model introduced in [1] and presented by system (1). When $\tau_1 = \tau_2$ and $I(Y, K) = I(Y) + qK$, we obtain the model presented and investigated in [8–13]. Further, the business cycle model presented by Wu and Wang in [14] is a special case of our model (2) when $\tau_1 = 0$ and $I(Y, K) = I(Y) + qK$.

The rest of this paper is organized as follows. The next section deals with some preliminary results about the well posedness and equilibria of system (2). In Section 3, we investigate the local stability of economic equilibrium and the existence of Hopf bifurcation. In Section 4, we establish the direction of Hopf bifurcation and the stability of bifurcating periodic solutions by applying the normal form method and center manifold theory. A special case and some numerical simulations are given in the last section.

2. Preliminary results

Let $\tau = \max\{\tau_1, \tau_2\}$ and $C = C([-\tau, 0], \mathbb{R}^2)$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^2$ equipped with the sup-norm $\|\varphi\| = \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|$ for $\varphi \in C$. As in

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[1], we assume that the general investment function $I(Y, K)$ satisfies the following hypothesis:

(H_1) There exist two constants $L > 0$ and $\bar{q} \geq 0$ such that $|I(Y, K) + \bar{q}K| \leq L$ for all $Y, K \in \mathbb{R}$.

Therefore, we get the following result.

Theorem 2.1. *Assume (H_1) holds. For any initial condition (ϕ_1, ϕ_2) $\in C$, there exists a unique solution of system (2) defined on $[0, +\infty)$ and this solution is uniformly bounded.*

Proof. By the standard theory of functional differential equations [15], we know that for any initial condition (ϕ_1, ϕ_2) $\in C$, there exists a unique local solution of system (2) on $[0, T_{max})$, where T_{max} is the maximal existence time for solution of system (2).

From the second equation of system (2), we have

$$K(t) = e^{-(\delta+\bar{q})t}\phi_2(0) + \int_0^t e^{-(\delta+\bar{q})(t-\xi)}[I(Y(\xi-\tau_1), K(\xi-\tau_2)) + \bar{q}K(\xi)]d\xi. \tag{3}$$

By using the assumption (H_1), it follows that

$$\begin{aligned}|K(t)| &\leq e^{-(\delta+\bar{q})t}|\phi_2(0)| + L\int_0^t e^{-(\delta+\bar{q})(t-\xi)}d\xi\\ &\leq e^{-(\delta+\bar{q})t}|\phi_2(0)| + \frac{L}{\delta+\bar{q}}.\end{aligned}$$

As $\lim_{t\to+\infty} e^{-(\delta+\bar{q})t}|\phi_2(0)| = 0$, then there exists a $t_0 > 0$ such that $e^{-(\delta+\bar{q})t}|\phi_2(0)| \leq 1$ for all $t \in [t_0, T_{\max})$. Thus, $|K(t)| \leq 1 + \frac{L}{\delta+\bar{q}}$, which implies that $K(t)$ is uniformly bounded for bound $A_1 = 1 + \frac{L}{\delta+\bar{q}}$.

Now, we show the uniform boundedness of Y. According to the first equation of system (2), we get

$$Y(t) = Y(t_0)e^{-\alpha\gamma(t-t_0)} + \alpha\int_{t_0}^t e^{-\alpha\gamma(t-\xi)})I(Y(\xi), K(\xi))d\xi, \ t \geq t_0. \tag{4}$$

Thus,

$$\begin{aligned}|Y(t)| &\leq |Y(t_0)|e^{-\alpha\gamma(t-t_0)} + \alpha\int_{t_0}^t e^{-\alpha\gamma(t-\xi)}\big(L + \bar{q}|K(\xi)|\big)d\xi\\ &\leq |Y(t_0)|e^{-\alpha\gamma(t-t_0)} + \frac{L+\bar{q}A_1}{\gamma}.\end{aligned}$$

Analogously to above, we deduce that the function $Y(t)$ is uniformly bounded for bound $A_2 = 1 + \frac{L+\bar{q}A_1}{\gamma}$.

By the above and using the continuity of the functions $Y(t)$ and $K(t)$, we conclude that $Y(t)$ and $K(t)$ are uniformly bounded $[0, T_{max})$. Consequently, $T_{max} = +\infty$. □

In the order to investigate the existence of equilibria of (2), we consider the following hypotheses:

(H_2) $I(0, 0) > 0$;
(H_3) $\frac{\partial I}{\partial Y}(Y, K) - \gamma < -\frac{\gamma}{\delta}\frac{\partial I}{\partial K}(Y, K)$ for all $(Y, K) \in \mathbb{R}^2$.

By using the same technique in [1], it is not hard to get the following result.

Theorem 2.2. *If (H_1) – (H_3) hold, then system (2) has a unique economic equilibrium of the form $E^*(Y^*, \frac{\gamma}{\delta}Y^*)$, where Y^* is the positive solution of the equation $I(Y, \frac{\gamma}{\delta}Y) - \gamma Y = 0$.*

3. Stability analysis and Hopf bifurcation

In this section, we focus on the local stability of the economic equilibrium $E^*(Y^*, K^*)$ and the existence of Hopf bifurcation.

Let $y = Y - Y^*$ and $k = K - K^*$. By substituting y and k into system (2) and linearizing, we get the following system

$$\begin{cases}\frac{dy}{dt} = \alpha[ay(t) + \beta k(t) - \gamma y(t)],\\ \frac{dk}{dt} = ay(t-\tau_1) + \beta k(t-\tau_2) - \delta k(t),\end{cases} \tag{5}$$

where $a = \frac{\partial I}{\partial Y}(Y^*, K^*)$ and $\beta = \frac{\partial I}{\partial K}(Y^*, K^*)$.

Hence, the characteristic equation about E^* is given by

$$\lambda^2 - \big[\alpha(a-\gamma) - \delta\big]\lambda - \alpha\beta a e^{-\lambda\tau_1} + \beta\big[\alpha(a-\gamma) - \lambda\big]e^{-\lambda\tau_2} - \alpha\delta(a-\gamma) = 0. \tag{6}$$

Similarly to the work of Zhou and Li [7], we distinguish three cases.

3.1. The case $\tau_1 = \tau_2 = 0$

When $\tau_1 = \tau_2 = 0$, Eq. (6) becomes

$$\lambda^2 - \lambda[\alpha(a-\gamma) + \beta - \delta] + \alpha(a-\gamma)(\beta-\delta) - \alpha\beta a = 0. \tag{7}$$

All roots of Eq. (7) have negative real parts if and only if

$$a - \gamma < \min\left\{\frac{\delta-\beta}{\alpha}, \frac{-a\beta}{\delta-\beta}\right\}. \tag{8}$$

Then the economic equilibrium E^* is locally asymptotically stable when (8) holds.

3.2. The case $\tau_1 \neq 0$, $\tau_2 = 0$

In this case, Eq. (6) takes the following form

$$\lambda^2 - \big[\alpha(a-\gamma) + \beta - \delta\big]\lambda + \alpha(a-\gamma)(\beta-\delta) - \alpha\beta a e^{-\lambda\tau_1} = 0. \tag{9}$$

Let $i\omega$ ($\omega > 0$) be a root of (9). So, we have

$$\begin{cases}-\omega^2 + \alpha(a-\gamma)(\beta-\delta) = \alpha a\beta\cos(\omega\tau_1),\\ -\omega[\alpha(a-\gamma) + \beta - \delta] = \alpha a\beta\sin(\omega\tau_1),\end{cases} \tag{10}$$

which leads to

$$\omega^4 + [\alpha^2(a-\gamma)^2 + (\beta-\delta)^2]\omega^2 + \alpha^2[(a-\gamma)^2(\beta-\delta)^2 - a^2\beta^2] = 0. \tag{11}$$

Let $u = \omega^2$. Hence, Eq. (11) becomes

$$F(u) = u^2 + [\alpha^2(a-\gamma)^2 + (\beta-\delta)^2]u + \alpha^2[(a-\gamma)^2(\beta-\delta)^2 - a^2\beta^2] = 0.$$

Since $\alpha^2(a-\gamma)^2 + (\beta-\delta)^2 > 0$, we have the following lemma.

Lemma 3.1.

(i) *If $|a-\gamma|(\delta-\beta) \geq -a\beta$, then Eq. (11) has no positive root.*
(ii) *If $|a-\gamma|(\delta-\beta) < -a\beta$, then Eq. (11) has a unique positive root given by*

$$\omega_0 = \frac{\sqrt{2}}{2}\Big(\sqrt{\Delta} - \alpha^2(a-\gamma)^2 - (\beta-\delta)^2\Big)^{1/2}, \tag{12}$$

where $\Delta = [\alpha^2(a-\gamma)^2 + (\beta-\delta)^2]^2 - 4\alpha^2[(a-\gamma)^2(\beta-\delta)^2 - a^2\beta^2]$.

From the above analysis and [1], we get the following results.