



The Ergodic Theorem for a new kind of attractor of a GIFS

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ABSTRACT

In 1987, Elton [11], has proved the first fundamental result on the convergence of IFS, the Elton's Ergodic Theorem. In this work we prove the natural extension of this theorem to the projected Hutchinson measure μ_α associated to a GIFSpdp $\mathcal{S} = (X, (\phi_j : X^m \rightarrow X)_{j=0,1,\dots,n-1}, (p_j)_{j=0,1,\dots,n-1})$, in a compact metric space (X, d) . More precisely, the average along of the trajectories $x_n(a)$ of the GIFS, starting in any initial points $x_0, \dots, x_{m-1} \in X$ satisfies, for any $f \in C(X, \mathbb{R})$,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n(a)) = \int_X f(t) d\mu_\alpha(t),$$

for almost all $a \in \Omega = \{0, 1, \dots, n-1\}^{\mathbb{N}}$, the symbolic space. Additionally, we give some examples and applications to Chaos Games and Nonautonomous Dynamical Systems defined by finite difference equations.

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Introduction

In 2008, Mihail and Miculescu [17], has introduced the Generalized Iterated Function Systems (GIFS, for short). They prove that there exists a fractal attractor and give estimates for the rate of convergence for contractive GIFS. In 2009, Mihail [16], has considered the Hutchinson measure associated to a GIFS with place dependent probabilities (GIFSpdp for short) that generalizes the classical Hutchinson measure, the invariant measure, associated to an Iterated Function System (IFS for short).

In this work, the central idea is to extend the GIFS to an IFS in a bigger space, that we call *Extended GIFS*. For this IFS we obtain an *Extended Hutchinson measure*. We also set up the properties of the extended Hutchinson measure and its relation with the original GIFS.

Using the extension, we prove an ergodic theorem that extends the classic Elton's ergodic theorem for IFS (Elton [11]) to the projected Hutchinson measure associated to a GIFS. From our results, we prove a Chaos game theorem for GIFS. As an application, we obtain some results on the stability and the asymptotic behavior of nonautonomous dynamical systems defined by finite difference equations. Also, we show how to recover properties of Gibbs measures for Hölder potentials through an appropriate GIFS that is built from an expansive endomorphism.

The paper has three sections. In the [Section 2](#), we recall the basic facts about GIFS and GIFSpdp, that we will use in the rest of the paper. In the [Section 3](#), we introduce the extension of a GIFS and prove the Ergodic Theorem ([Theorem 23](#)). [Section 4](#), is devoted to applications. The main goal here is to prove the Chaos Game Theorem ([Theorem 25](#)), allowing us to draw the attractor of the extended GIFS and his projection.

We believe that our results and technics will be useful for forthcoming works and for other researchers in this area. The ergodic theorem we prove and his consequences, provides a useful tool on the understanding of GIFS.

1. Motivation

A well known example of Discrete Nonautonomous Dynamical System (see Pötzsche [18]) is the dynamics generated by a finite difference equation(FDE), defined by a nonautonomous recurrence relation of order $m \geq 2$, nominally

$$x_{j+m} = f(x_{j+m-1}, x_{j+m-2}, \dots, x_j, \omega_j), \quad j \geq 0,$$

where $\omega_0, \omega_1, \dots \in I$. The control sequence $(\omega_j)_{j=0,1,\dots}$ represents some seasonal interference acting on each iteration by changing the standard recursiveness on a FDE. It can be modeled by a GIFS $\mathcal{S} = (X^m, (\phi_j)_{j=0,\dots,n-1})$ where $\phi_j: X^m \rightarrow X$ is given by

$$\phi_j(y_0, y_1, \dots, y_{m-1}) = f(y_0, y_1, \dots, y_{m-1}, \omega_j).$$

The control set I could be finite or not. In this case, the orbit of the GIFS from the point $(c_0, \dots, c_{m-2}, c_{m-1})$ is equal to the orbit of the

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FDE with initial conditions:

$$\begin{cases} x_{j+m} = f(x_{j+m-1}, x_{j+m-2}, \dots, x_j, \omega_j), \omega \in I^{\mathbb{N}} \\ x_0 = c_0 \\ \dots = \dots \\ x_{m-1} = c_{m-1} \end{cases}$$

If the associated GIFS satisfy the hypothesis E1 and E2, that we define in the next, we can apply our theory to study the asymptotic behavior and the limit sets of these nonautonomous dynamical systems.

2. Background on GIFS and GIFSpdp

In this section, we will recall the basic definitions and results on the theory of GIFS. See Miculescu [15], for more details and the notation. We notice that the word “generalized” has been used in several different ways representing more general components of a classical IFS. Here, generalized, means that functions are from X^m to X instead X to X .

2.1. Generalized iteration function system (GIFS)

Let (X, d) be a compact¹ metric space (typically $X = [0, 1]$, $X = \{0, 1, \dots, d-1\}^{\mathbb{N}}$, $X = [0, 1]^{\mathbb{N}}$, etc). Consider the topology on X^m given by

$$d_{\infty}((x_1, x_2, \dots, x_m), (y_1, y_2, \dots, y_m)) = \max_{i=1, \dots, m} d(x_i, y_i),$$

then (X^m, d_{∞}) is also a compact metric space.

Definition 1. A (continuous) generalized iterated function system (GIFS) of degree m is a (finite) family $\mathcal{S}$ of continuous functions $\phi_j: X^m \rightarrow X$, denoted by $\mathcal{S} = (X, (\phi_j)_{j=0, \dots, n-1})$.

See Secelean [19,20] for the analogous theory for countable GIFS. In order to avoid technicalities we assume that $m = n = 2$ that is, two maps in X^2 (see Remark 6 for additional details). So the standard GIFS is $\mathcal{S} = (X, (\phi_j)_{j=0,1})$ where $\phi_0, \phi_1: X^2 \rightarrow X$. We recall that,

$$\begin{aligned} Lip(X^2, X) &= \{f \mid d(f(x_0, y_0), f(x_1, y_1)) \\ &\leq C d((x_0, y_0), (x_1, y_1)), C := Lip(f)\} \end{aligned}$$

and

$$\begin{aligned} Lip_{C,D}(X^2, X) &= \{f \mid d(f(x_0, y_0), f(x_1, y_1)) \\ &\leq C d(x_0, x_1) + D d(y_0, y_1), C, D > 0\}. \end{aligned}$$

From now on we will assume the contraction hypothesis:

E1 - Each $\phi_j: X^2 \rightarrow X$ is in $Lip_{C_j, D_j}(X^2, X)$ and $C_j + D_j < 1$. In particular, all the ϕ_j are Lipschitz contractions and $Lip(\phi_j) = C_j + D_j$.

As usual, we denote $\mathcal{K}(X) \subset \mathcal{P}(X) = 2^X$, the family of compact subsets of X . Moreover, $\mathcal{K}^*(X) = \mathcal{K}(X) \setminus \{\emptyset\}$.

Definition 2. Given $f: X^2 \rightarrow X$ we define the associated set function $F_f: \mathcal{P}^*(X)^2 \rightarrow \mathcal{P}^*(X)$ by $F_f(K_1, K_2) = f(K_1 \times K_2)$. Also, we define the function $F_{\mathcal{S}}: \mathcal{K}^*(X)^2 \rightarrow \mathcal{K}^*(X)$ associated to $\mathcal{S}$ by $F_{\mathcal{S}}(K_1, K_2) = \bigcup_{j=0,1} F_{\phi_j}(K_1, K_2)$. A set $Y \subseteq X$ is self-similar (or fractal) with respect to $\mathcal{S}$ if $F_{\mathcal{S}}(Y, Y) = Y$.

The map $F_{\mathcal{S}}$ is sometimes called, Fractal operator, Barnsley's Function or Hutchinson's operator, in the literature. From Mihail and Miculescu [17], Theorem 3.5, we know that under the hypothesis E1, there exists a unique attractor $A(\mathcal{S}) \in \mathcal{K}^*(X)$ for the

GIFS that depends continuously on ϕ_j . That is, $A(\mathcal{S})$ is self-similar ($F_{\mathcal{S}}(A(\mathcal{S}), A(\mathcal{S})) = A(\mathcal{S})$) and, for any $H_0, H_1 \in \mathcal{K}^*(X)$ the recursive sequence of compact subsets $H_{j+2} = F_{\mathcal{S}}(H_{j+1}, H_j)$, converges to $A(\mathcal{S})$ with respect to the Hausdorff metric: $A(\mathcal{S}) = \lim_{j \rightarrow \infty} H_j$.

The natural question to make about GIFS is if they offer some new fractals. The positive answer is given by Mihail and Miculescu [17] through examples and in the recent work Strobin [22] for a more general case. We will discuss that, in the end of the Section 3.1. We should mention that recently, in 2015, Dumitru et al. [9] they consider many questions regarding to the extension of the concept of GIFS for topological contractions assuming that the family of maps is not just finite or countable but possibly an arbitrary family $\mathcal{F}$ of maps from X^m to X , satisfying suitable hypothesis. Several results are obtained by using code spaces (see [21] for details).

2.2. GIFS with place dependent probabilities (GIFSpdp)

In this section, we use the notation in Miculescu [15]. The set $Prob(X)$ will always be the set of regular Borel probabilities on X with respect to the Borel sigma algebra induced by the metric.

Definition 3. A generalized iteration function system with place dependent probabilities (GIFSpdp) is a family $\mathcal{S}$ of continuous functions $\phi_j: X^2 \rightarrow X$, and weight functions (probabilities) $p_j: X^2 \rightarrow [0, 1]$ such that $p_0(x, y) + p_1(x, y) = 1$, denoted $\mathcal{S} = (X, (\phi_j)_{j=0,1}, (p_j)_{j=0,1})$.

One special case is when the probabilities are given by a potential function $u: X \rightarrow \mathbb{R}$, then $p_j(x, y) = u(\phi_j(x, y))$ and $u(\phi_0(x, y)) + u(\phi_1(x, y)) = 1$. We denote such case as a uniform GIFSpdp according to Lopes and Oliveira [13]. We consider an application of this after Theorem 28.

E2 - For a GIFSpdp we assume two hypothesis on the weights:

- $p_i(x, y) \geq \delta > 0$, for any $i = 0, 1$, $x, y \in X$;
- $p_i(x, y)$ is in $Lip_{c_i, d_i}(X^2, [0, 1])$ with $c_i + d_i < 1$.

We recall that $p_i(x, y)$ is Dini continuous if $\int_0^\varepsilon \frac{Q_i(t)}{t} dt < \infty$ for some $\varepsilon > 0$, where Q_i is the modulus of continuity of p_i , $|p_i(x, y) - p_i(x', y')| \leq Q_i(d((x, y), (x', y')))$, $\forall (x, y) \neq (x', y')$. For instance if p_i is β -Hölder ($Q_i(t) = kt^\beta$) or p_i is k -Lipschitz ($Q_i(t) = kt$) then $p_i(x, y)$ is Dini continuous.

Definition 4. Given $\mathcal{S} = (X, (\phi_j)_{j=0,1}, (p_j)_{j=0,1})$ we define (see Miculescu [15]), the transfer operator $B_{\mathcal{S}}: C(X, \mathbb{R}) \rightarrow C(X^2, \mathbb{R})$ by

$$B_{\mathcal{S}}(f)(x, y) = \sum_{j=0,1} p_j(x, y) f(\phi_j(x, y)),$$

for all $(x, y) \in X^2$. The Markov operator $\mathcal{L}_{\mathcal{S}}: Prob(X) \times Prob(X) \rightarrow Prob(X)$ is given by

$$\int_X f(t) d\mathcal{L}_{\mathcal{S}}(\mu, \nu)(t) = \int_{X^2} B_{\mathcal{S}}(f)(x, y) d(\mu \times \nu)(x, y),$$

for any $\mu, \nu \in Prob(X)$ and any continuous $f: X \rightarrow \mathbb{R}$.

We recall that, under the hypothesis E1 and E2 we get, from Miculescu [15], Theorem 4.4, that:

- There is a unique $\mu_{\mathcal{S}} \in Prob(X)$ such that $\mathcal{L}_{\mathcal{S}}(\mu_{\mathcal{S}}, \mu_{\mathcal{S}}) = \mu_{\mathcal{S}}$;
- $\text{supp}(\mu_{\mathcal{S}}) = A(\mathcal{S})$, the attractor of the GIFS;
- For any $\mu_0, \mu_1 \in Prob(X)$ the sequence $\mu_{j+2} = \mathcal{L}_{\mathcal{S}}(\mu_j, \mu_{j+1})$ converges in the Monge-Kantorovich distance d_H^2 (see [12], Definition 2.53), to $\mu_{\mathcal{S}}$.

¹ We assume compactness to avoid technicalities. Many of the results that we present here are true if (X, d) is just complete. It is sufficient because the measures we use are always supported in the attractors that are compact sets.

² $d_H(\mu, \nu) = \sup_{Lip(f) \leq 1} \int f d\mu - \int f d\nu$, for any $\mu, \nu \in Prob(X)$.

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