Chaos, Solitons and Fractals 98 (2017) 145–151

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Effect of topological structure on synchronizability of network with connection delay

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ARTICLE INFO

Article history:
Received 13 January 2016
Revised 11 March 2017
Accepted 13 March 2017

MSC:
34K18
92K20

Keywords:
Synchronization
Time delay
Master stability function

ABSTRACT

The effect of topological structure on synchronizability of network with connection delay is discussed in this paper. By introducing a new time variable in the master stability function, it is shown the effect of connection delay can be weakened when the maximum absolute value of the eigenvalues corresponding to the transverse directions is small. And then, on the basis of the results of local stability of time-delayed system, it is indicated that the network can achieve complete synchronization in larger region of the relevant bifurcation parameters when its outer-coupling matrix is with smaller maximum absolute value of the eigenvalues corresponding to the transverse directions. Numerical studies demonstrate the analytical finding.

1. Introduction

Coupled systems will adjust their behaviors with each other, such that there are some relations between their dynamical behaviors, this phenomenon is called synchronization. The phenomenon of synchronization has drawn much attention from researchers and engineers since it was first reported in the study of two coupled pendulums [1]. Various kinds of synchronization have been proposed in the literature, such as complete synchronization, phase synchronization, generalized synchronization, lag synchronization, and so on [2,3]. Complete synchronization is the most strict one, which means all the coupled systems do the same things at the same time. A striking finding is that complete synchronization can occur in coupled chaotic systems [4,5], and that phenomenon has found wide applications in engineering [6]. To describe the local stability of synchronization manifold, condition Lyapunov exponent is introduced by Pecora, Carroll, Johnson and Mar [7], and the synchronization manifold is locally stable if the largest condition Lyapunov exponent is negative, and thus, the coupled systems can get complete synchronization. Global synchronization is usually studied based on the Lyapunov function method [8,9], when the synchronization manifold is globally stable, the coupled systems can get complete synchronization under arbitrarily given initial conditions [5,10].

Time delay should be considered in coupled systems due to finite information transmission and processing speed. Studies indicate time delay has significant effect on characteristic of synchronization. Time delay can lead to amplitude death of two coupled limit cycle oscillators even if they have the same frequency, this is in sharp contrast to the situation without time-delay, where the amplitude death can only happened when their frequency is sufficiently disparate [11]. Cluster synchronization in nearest-neighbor delay-coupled limit-cycle oscillators was considered in [12], it is shown time delay has an important effect on the stability of various cluster states, and the mean frequency is decreasing with increase of the time delay. Time delay is also included in the synchronized system which describes the dynamics of synchronized state on the synchronization manifold, thus, time-delay can change the dynamics of synchronized state, and leads to complex synchronized dynamics [13,14]. Delay-induced various transitions of synchronization are observed [15,16]. Though time delay frequently induces complex synchronized dynamics and synchronization transitions in coupled systems, under certain conditions, time-delay can be a positive factor for synchronization [17]. Based on the plot of largest condition Lyapunov exponent obtained in [18], it is shown the complete synchronization occurs with very low coupling strength when time delay is large.

Real network has various topological structures, such as regular network, random network, small-world network, scale-free network, switching-topology network and etc. [19,20]. Network topology plays a crucial role in synchronizability. To study network-topology effects, an universal master stability function is

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http://dx.doi.org/10.1016/j.chaos.2017.03.035
0960-0779/© 2017 Published by Elsevier Ltd.

introduced by separating the synchronization manifold direction from other transverse directions [21], and the network can get synchronized when all the largest condition Lyapunov exponents, which are governed by the master stability function, are negative, and thus, the synchronizability of network is related to the eigenvalues of outer-coupling matrix of the network. Some fully-coupled networks can get synchronization with any small single-edge coupling strength only if the network scale is large enough [19]. In nearest-neighbor coupled network, synchronizability is always decreasing with the increase of network scale, however, its synchronizability can be enhanced significantly when a few edges are randomly added in the network [22]. For scale-free network, it is shown its synchronizability will increase and saturate with the increase of network scale [23].

When connection delay is involved in network, the effect of network topology is different to the case without connection delay due to the fact that time-delay can lead to stability switch in master stability function and result to dynamic transition in synchronized system. The master stability function of delay-coupled systems was derived by directly transforming the master stability function of non-delay-coupled systems [18], and was further generalized to the case of distributed-delay-coupled systems [24]. On the basis of master stability function, the synchronization of delay-coupled FitzHugh-Nagumo oscillators was studied in [25], it is shown superimposing inhibitory links randomly on top of a regular ring of excitatory coupling leads to a phase transition from synchronization to asynchronization. The cluster and group synchronization in delay-coupled networks was investigated in [26], the studies indicate the master stability function has a discrete rotational symmetry depending on the number of groups.

This paper presents an investigation of network-topology effect on the synchronizability of network with connection delay, where the synchronizability is characterized by the synchronized region of relevant bifurcation parameters, and the synchronizability of a network is said to be stronger if this network can get synchronization in larger region of relevant bifurcation parameters. In next section, the master stability function for coupled systems with connection delay is derived. Then in Section 3, the effect of network topology on synchronizability of delay-coupled systems is studied, and case studies are given to demonstrate the analytical finding. Some conclusions are drawn in the last section.

2. Master stability function

Consider complete synchronization of the following delay-coupled network with n identical nodes

$$\dot{x}_i = F(x_i) + e\,a_{ii}\,H(x_i) + e\sum_{j=1,j\neq i}^{n} a_{ij}\,H(x_j(t-\tau)), \quad i = 1, 2, \cdots, n \tag{1}$$

where a dot represents derivative with respect to time t, $\dot{x}_i = F(x_i)$ describes the dynamics of a single node, $x_i = (x_{i1}, x_{i2}, \cdots, x_{im})^T \in \mathrm{R}^m$, $F(\cdot) = (f_1(\cdot), f_2(\cdot), \cdots, f_m(\cdot))^T \in \mathrm{R}^m$ is Lipschitz, e is the single-edge inputting strength, τ represents connection delay, $H(\cdot) = (h_1(\cdot), h_2(\cdot), \cdots, h_m(\cdot))^T \in \mathrm{R}^m$ is inner coupling function, and $A = \{a_{ij}\}_{n\times n}$ is the outer-coupling matrix.

To obtain the master stability function for Eq. (1), introduce the general hypotheses that $\sum_{j=1}^{n} a_{ij} = 0$ $(i = 1, 2, \cdots, n)$ with $a_{ii} = -k_0$ and $a_{ij} = 1$ or 0 $(i \neq j)$, where k_0 is actually the input degree of each node in the network. Many typical network topologies are covered by Eq. (1) including the fully-coupled network and locally-coupled network.

Dhamala, Jirsa and Ding [18] have derived the master stability function of delay-coupled systems by directly transforming the master stability function of non-delay-coupled systems derived by Pecora and Carroll [21]. Here, we derive the master stability function of Eq. (1) in a heuristic way by separating the synchronization manifold direction from other transverse directions as done for non-delay-coupled systems [21] and distributed-delay-coupled systems [24].

In the synchronization manifold, the dynamics of synchronized state $x_1 = x_2 = \cdots = x_n = u$ is described as

$$\dot{u} = F(u) + e\,k_0\,(H(u(t-\tau)) - H(u)) \tag{2}$$

To study the stability of synchronization manifold, consider the local dynamics of perturbed variables $\delta_i = x_i - u$ $(i = 1, 2, \cdots, n)$, which are governed by the following linearized equations

$$\dot{\delta}_i = \mathrm{D}F\,\delta_i - e\,k_0\,\mathrm{D}H\,\delta_i + e\,\mathrm{D}H_\tau \sum_{j=1,j\neq i}^{n} a_{ij}\,\delta_j(t-\tau), \quad i = 1, 2, \cdots, n \tag{3}$$

where DF is the Jacobian matrix of $F(\cdot)$ evaluated on the synchronized state $u(t)$, DH and DH_τ are the Jacobian matrix of $H(\cdot)$ evaluated on the synchronized state $u(t)$ and delayed synchronized state $u(t-\tau)$ respectively. Let $\Delta = (\delta_1, \delta_2, \cdots, \delta_n) \in \mathrm{R}^{m\times n}$ and $B = (A - \mathrm{diag}\{a_{ii}\})^{\mathrm{T}}$, then one has

$$\dot{\Delta} = \mathrm{D}F\,\Delta - e\,k_0\,\mathrm{D}H\,\Delta + e\,\mathrm{D}H_\tau\,\Delta(t-\tau)\,B \tag{4}$$

Diagonalize matrix B as

$$B = Q\,\Lambda\,Q^{-1}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, λ_i $(i = 1, 2, \cdots, n)$ are the eigenvalues of B with $\lambda_1 = k_0$, $Q = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, ζ_i $(i = 1, 2, \cdots, n)$ are the eigenvectors of B with respective to λ_i with $\zeta_1 = (1, 1, \cdots, 1)^T$, and ζ_i $(i = 1, 2, \cdots, n)$ are orthogonal to each other. Introduce new perturbed variables $\eta_i = \Delta\,\zeta_i$ $(i = 1, 2, \cdots, n)$, one has the following equations

$$\dot{\eta}_1 = \mathrm{D}F\,\eta_1 + e\,k_0\,(\mathrm{D}H_\tau\,\eta_1(t-\tau) - \mathrm{D}H\,\eta_1) \tag{5}$$

and

$$\dot{\eta}_i = \mathrm{D}F\,\eta_i - e\,k_0\,\mathrm{D}H\,\eta_i + e\,\lambda_i\,\mathrm{D}H_\tau\,\eta_i(t-\tau), \quad i = 2, 3, \cdots, n \tag{6}$$

Obviously, Eq. (5) is the perturbed equation on the direction of synchronization manifold, and Eq. (6) are the perturbed equations on the transverse directions, and λ_i $(i = 2, 3, \cdots, n)$ are the eigenvalues corresponding to the transverse directions. The synchronization manifold is stable when all the largest condition Lyapunov exponents governed by Eq. (6) are negative.

Note that Eq. (6) are with the same structure, which leads to the master stability function

$$\dot{\xi} = \mathrm{D}F\,\xi - e\,k_0\,\mathrm{D}H\,\xi + e\,\lambda\,\mathrm{D}H_\tau\,\xi(t-\tau) \tag{7}$$

The region of the relevant bifurcation parameters that guarantees the largest condition Lyapunov exponent of Eq. (7) be negative is called synchronized region, and the synchronization manifold is stable when the relevant bifurcation parameters fall into the synchronized region.

3. Network-topology effect

Let $C = e\,k_0$ denote the total inputting strength of each node, and take C as one of the system parameters, then Eq. (7) becomes

$$\dot{\xi} = \mathrm{D}F\,\xi - C\,\mathrm{D}H\,\xi + \frac{C}{k_0}\,\lambda\,\mathrm{D}H_\tau\,\xi(t-\tau) \tag{8}$$