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On Hopf bifurcation in fractional dynamical systems

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ABSTRACT

Fractional order dynamical systems admit chaotic solutions and the chaos disappears when the fractional order is reduced below a threshold value [1]. Thus the order of the dynamical system acts as a chaos controlling parameter. Hence it is important to study the fractional order dynamical systems and chaos. Study of fractional order dynamical systems is still in its infancy and many aspects are yet to be explored.

In pursuance to this in the present paper we prove the existence of fractional Hopf bifurcation in case of fractional version of a chaotic system introduced by Bhalekar and Daftardar-Gejji [2]. We numerically explore the (A, B, α) parameter space and identify the regions in which the system is chaotic. Further we find (global) threshold value of fractional order α below which the chaos in the system disappears regardless of values of system parameters A and B.

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1. Introduction

Fractional order dynamical systems are gaining popularity due to their widespread applications [3]. The study of chaotic dynamical systems of fractional order was initiated by Grigorenko and Grigorenko [1] wherein fractional ordered Lorenz system was studied. It was shown that the order of the derivative acts as a chaos controlling parameter and below a threshold value of α, chaos disappears. These simulations were done by keeping rest of the system parameters fixed. Since then there has been increasing interest in this topic and a large number of contributions have appeared in the literature which deal with fractional versions of various chaotic systems including Chen System [4], Rössler system [5], Liu system [6], financial system [7] and so on [3].

In spite of the extensive numerical work, our understanding of fractional systems is not complete and very few analytical results have been obtained. The first important result obtained regarding stability analysis of fractional systems is due to Matignon [8]. Some of the important results regarding stability of fractional systems have been summarized by Li and Zhang [9].

The system introduced by Bhalekar and Daftardar-Gejji (BG system) has been shown to be chaotic for certain values of parameters [2]. Further the forming mechanism of this system is discussed by Bhalekar [10]. The synchronization and anti-synchronization of Bhalekar – Gejji system and Liu system is done by Singh et al. [11]. The Hopf bifurcation in integer order Bhalekar – Gejji system has been explored by Aqeel and Ahmad [12]. In the present paper we prove existence of Hopf bifurcation in fractional version of BG system and explore the parameter space numerically.

The paper is organized as follows. Section 2 comprises of preliminaries and notations. Section 3 deals with fractional Hopf bifurcation and proves its existence for fractional BG system. Sections 4, 5 and Section 6 contain numerical explorations for various system parameters. Conclusions are summarized in Section 7.

2. Preliminaries

In this section, we introduce notations, definitions and preliminaries pertaining to fractional calculus and stability of fractional dynamical systems [1,9,13,14]. $N_r(a)$ denotes the neighborhood of point $a \in \mathbb{R}^n$ having radius $r > 0$. $\|.\|$ denotes standard Euclidean norm on $\mathbb{R}^n$.

Definition 1 [15]**.** The fractional integral of order $\alpha > 0$ of a real valued function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau.$$

Definition 2 [15]**.** Caputo fractional derivative of order $\alpha > 0$ of a real valued function f is defined as

$$\begin{aligned} D^\alpha f(t) &= I^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad m-1 < \alpha < m, \\ &= f^{(m)}(t), \quad \alpha = m, \ m \in \mathbb{N}. \end{aligned}$$

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We assume $0 < \alpha \leq 1$throughout the paper.

Consider the fractional order autonomous system with bifurcation parameter $\mu \in \mathbb{R}^m$

$$D^\alpha x(t) = f_\mu(x), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{1}$$

Definition 3 [9]**.** A point $e_\mu \in \mathbb{R}^n$ is called as an **equilibrium point** of (1) if $f_\mu(e_\mu) = 0$.

Definition 4 [9]**.** The system (1) is called as **locally stable** if for every $\epsilon > 0, \exists \delta > 0$ such that $x_0 \in N_\delta(e_\mu) \Rightarrow \|x_\mu(t) - x_\mu(e_\mu)\| < \epsilon$ for $t > 0$.

Definition 5 [9]**.** The system (1) is called as (locally) **asymptotically stable** if for $x_\mu(t)$ as above, $\|x_\mu(t) - x_\mu(e_\mu)\| \to 0$ as $t \to \infty$.

A system is **unstable** if it is not stable.

The linearization of (1) around the equilibrium point e_μ is given as

$$D^\alpha x(t) = Jx(t), x(0) = x_0, \tag{2}$$

where $[J]_{i,j} = \frac{\partial f^i_\mu}{\partial x_j}(e_\mu), \ 1 \leq i, j \leq n$. Clearly eigenvalues λ of J depend on the bifurcation parameter. To simplify the notation we generally drop the suffix μ for λ unless it needs to be emphasized.

Definition 6 [9]**.** The linearized system (2) is called as (locally) **linearly stable** if for each eigenvalue λ of J, $|arg(\lambda)| > \frac{\pi\alpha}{2}$.

The system (2) is said to be **linearly unstable** if $|arg(\lambda)| < \frac{\pi\alpha}{2}$, for at least one eigenvalue λ of J. The equilibrium point e_μ is defined as non-hyperbolic equilibrium point if $|arg(\lambda)| = \frac{\pi\alpha}{2}$, for some eigenvalue λ of J.

This stability criteria is due to Matignon [8], which coincides with the concept of local stability for fractional systems (1) [16].

Definition 7. Given that $\mu = \mu_0$ fixed, **threshold value** $\alpha^*_{\mu_0}$ for the system (1) is defined as

$$\alpha^*_{\mu_0} = \inf_{\alpha_c}\{\text{for } \alpha > \alpha_c, \text{system (1) is chaotic}\}. \tag{3}$$

Clearly for $\alpha < \alpha^*_{\mu_0}$ chaos in the system $D^\alpha x(t) = f_{\mu_0}(x)$ disappears.

In the next section we highlight system parameter dependence of this definition. Further we propose a definition of global threshold value for parameter μ.

3. Analysis of fractional BG system

The system is said to undergo a Hopf bifurcation when an equilibrium point switches the stability along-with creation or destruction of certain periodic orbits [13]. For the integer order system, this is known to occur when equilibrium has pair of eigenvalues that cross the imaginary axis at non-zero speed.

Due to the changed stability criteria for fractional systems [8], it is quite natural to frame the existence criteria for the fractional Hopf bifurcation as follows.

Existence criteria for Fractional Hopf Bifurcation. Consider the system of fractional differential equations given by Eq. (1) together with bifurcation parameter $\mu \in \mathbb{R}$ and $\alpha \in (0, 1]$. Let e_μ be the equilibrium point of (1) and (2) be its linearization around e_μ.

Suppose $n \times n$ matrix A has $\lambda_1(\mu)$, $\lambda_2(\mu)$,..., $\lambda_n(\mu)$ as its eigenvalues such that at least one pair of eigenvalues say $\lambda_1(\mu)$, $\lambda_2(\mu)$ is complex conjugate.

We say that (1) undergoes fractional Hopf bifurcation, if $\exists$ a critical value $\mu = \mu_h$ such that the following conditions are satisfied.

1. $\lambda_1(\mu_h)$ *and* $\lambda_2(\mu_h)$ *satisfy* $|\arg(\lambda_j(\mu_h))| = \frac{\pi\alpha}{2} \quad (j = 1, 2)$,
2. $|\arg(\lambda_i(\mu_h))| \neq \frac{\pi\alpha}{2}, \quad (i = 3, 4, \cdots, n)$,
3. $\frac{d}{d\mu} |\arg(\lambda_j(\mu))| \,|_{\mu=\mu_h} \neq 0, \quad (j = 1, 2)$.

First and second conditions are sometimes called as singularity conditions while the third is transversality condition.

Daftardar-Gejji and Bhalekar introduced a new dynamical system referred to as Bhalekar–Gejji (BG) system. This system exhibits chaos for certain parameter values [12]. In the present paper we investigate the fractional version of the system *i.e.*

$$D^\alpha x(t) = \omega x(t) - y^2(t), \tag{4}$$

$$D^\alpha y(t) = \mu(z(t) - y(t)), \tag{5}$$

$$D^\alpha z(t) = Ay(t) - Bz(t) + x(t)y(t), \tag{6}$$

where $0 < \alpha \leq 1$ and ω, μ, A, B are parameters. Standard values for which the system exhibits chaos are $\omega = -2.667, \ \mu = 10, \ A = 27.3, \ B = 1$. μ is generally taken as positive while ω is a negative real number.

Consider for further analysis $\omega < 0$ and $\mu > 0$. Objective of this paper is to show that Bhalekar–Gejji system satisfies the above criteria and hence the route taken by system to chaos is of fractional Hopf bifurcation. Further we extensively analyze the effect of variation of parameter in $A - B$ plane on the system. For fractional systems, fractional order α also acts as a bifurcation parameter. We study effect of variation of α over the system as well. As a result we identify the exact region in $A - B$ plane for which the system is stable and chaotic.

For $B - A > 0$, system has only one equilibrium point *i.e.* $P_1(0, 0, 0)$. At $B = A$ system undergoes **supercritical pitchfork bifurcation**, with origin turning into index 1 saddle point with formation of two new symmetrically opposite equilibrium points P_2, P_3. For $B - A \leq 0$, $P_2(B - A, \sqrt{\omega(B-A)}, \sqrt{\omega(B-A)})$ and $P_3(B - A, -\sqrt{\omega(B-A)}, -\sqrt{\omega(B-A)})$.

The Jacobian matrix of (4) is given as

$$J = \begin{pmatrix} \omega & -2y & 0 \\ 0 & -\mu & \mu \\ y & A+x & -B \end{pmatrix}. \tag{7}$$

Around origin $J|_{P_1}$ the eigenvalues are given as $\frac{1}{2}[-B - \mu \pm \sqrt{(B-\mu)^2 + 4A\mu}]$ and ω. Since $\omega < 0$, stability analysis as given above is easy to verify.

For P_2 and P_3, eigenvalues are roots of the same characteristic polynomial. Let $f(\lambda)$ denote the characteristic polynomial of P_2 and P_3. Then

$$f(\lambda) = \lambda^3 + (B + \mu - \omega)\lambda^2 - \omega(B + \mu)\lambda + 2\mu\omega(B - A) \tag{8}$$

and $f(\lambda) = 0$ is the corresponding characteristic equation. Fractional Hopf bifurcation will occur when complex roots of $f(\lambda)$ will cross into the cone $|arg(\lambda)| < \frac{\pi\alpha}{2}$. Set $A = A_h$, the Hopf critical value, in this case $\lambda = re^{i\theta}$ where $\theta = \pm\frac{\pi\alpha}{2}$ will satisfy $f(\lambda) = 0$. Thus we get

$$\begin{aligned} &r^3e^{i3\theta} + (B + \mu - \omega)r^2e^{2i\theta} - \omega(B + \mu)re^{i\theta} + 2\mu\omega(B - A_h) = 0 \\ &r^3(\cos(3\theta) + i\sin(3\theta)) + (B + \mu - \omega)r^2(\cos(2\theta) + i\sin(2\theta)) \\ &\quad -\omega(B + \mu)r(\cos(\theta) + i\sin(\theta)) + 2\mu\omega(B - A_h) = 0. \end{aligned} \tag{9}$$

Equating real and imaginary parts in (9) we get

$$\begin{aligned} &\cos(3\theta)r^3 + (B + \mu - \omega)\cos(2\theta)r^2 \\ &\quad -\omega(B + \mu)\cos(\theta)r + 2\mu\omega(B - A_h) = 0, \end{aligned} \tag{10}$$

$$\sin(3\theta)r^3 + (B + \mu - \omega)\sin(2\theta)r^2 - \omega(B + \mu)\sin(\theta)r = 0. \tag{11}$$

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