Chaos, Solitons and Fractals 98 (2017) 220–232

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Frontiers

Rate of convergence: the packing and centered Hausdorff measures of totally disconnected self-similar sets☆

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ARTICLE INFO

Article history:
Received 30 June 2016
Accepted 7 March 2017
Available online 28 March 2017

MSC:
Primary: 28A75
28A80

Keywords:
Packing measure
Centered Hausdorff measure
Self-similar sets
Computability of fractal measures
Rate of convergence

ABSTRACT

In this paper we obtain the rates of convergence of the algorithms given in [13] and [14] for an automatic computation of the centered Hausdorff and packing measures of a totally disconnected self-similar set. We evaluate these rates empirically through the numerical analysis of three standard classes of self-similar sets, namely, the families of Cantor type sets in the real line and the plane and the class of Sierpinski gaskets. For these three classes and for small contraction ratios, sharp bounds for the exact values of the corresponding measures are obtained and it is shown how these bounds automatically yield estimates of the corresponding measures, accurate in some cases to as many as 14 decimal places. In particular, the algorithms accurately recover the exact values of the measures in all cases in which these values are known by geometrical arguments. Positive results, which confirm some conjectural values given in [13] and [14] for the measures, are also obtained for an intermediate range of larger contraction ratios. We give an argument showing that, for this range of contraction ratios, the problem is inherently computational in the sense that any theoretical proof, such as those mentioned above, might be impossible, so that in these cases, our method is the only available approach. For contraction ratios close to those of the connected case our computational method becomes intractably time consuming, so the computation of the exact values of the packing and centered Hausdorff measures in the general case, with the open set condition, remains a challenging problem.

1. Introduction

The present paper is part of a program aimed at finding a method for the automatic computation of metric measures, such as the packing or Hausdorff measure, of a given fractal set. In particular, we obtain the rates of convergence of the algorithms given in [13] and [14] for computing the centered Hausdorff and packing measures, respectively, of a totally disconnected self-similar set. It is important to note that, although the convergence of these algorithms was shown in [13] and [14] without establishing their rates of convergence, the outputs of the algorithms were still useful for obtaining conjectural values of the measures. Using results presented in this note we can prove these conjectures (see Sections 2.3 and 3.3).

Recall that a totally disconnected self-similar set associated to a system $\Psi = \{f_1, f_2, \ldots, f_m\}$ of contracting similitudes of $\mathbb{R}^n$ is a compact non-empty set $E \subset \mathbb{R}^n$ such that $E = \bigcup_{i=1}^{m} f_i(E)$ and satisfying

$$f_i(E) \cap f_j(E) = \emptyset \quad \forall\, i \neq j, \qquad i, j \in \{1, \ldots, m\} =: M. \tag{1}$$

The last condition implies the *Open Set Condition (OSC, see [9])*, and it is known as the *Strong Separation Condition (SSC)*. Throughout the paper we assume the system Ψ satisfies the *SSC* and write

$$c := \min_{\substack{i,j \in M \\ i \neq j}} d_{\inf}(f_i(E), f_j(E)) > 0, \tag{2}$$

where $d_{\inf}(f_i(E), f_j(E))$ is the distance between $f_i(E)$ and $f_j(E)$. The *similarity ratio* of $f_i \in \Psi$ is denoted by $r_i \in (0, 1)$, and we write

$$r_{\min} := \min_{i=1,\ldots,m} r_i \qquad \text{and} \qquad r_{\max} := \max_{i=1,\ldots,m} r_i. \tag{3}$$

Both algorithms are based on the self-similar tiling principle stated in [17] for self-similar sets satisfying the OSC. In [17] it was shown that if B is any closed subset (or tile) of a self-similar set E such that $\mu(B) > 0$, where μ is the invariant measure (see (8)), then E can be tiled, without any loss of μ-measure, by a

☆ This research was supported by the Complutense University of Madrid and Santander Universidades, research project 940038.

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http://dx.doi.org/10.1016/j.chaos.2017.03.007
0960-0779/

countable collection of tiles that are images of B under similitudes. Recall from [9] that, for self-similar sets satisfying the OSC, the measure μ is a multiple of any scaling measure, and in particular of the packing, Hausdorff, or centered Hausdorff measure.

Taking an appropriate initial tile B (one with minimal spherical density in the case of the packing measure, and with maximal spherical density in the case of the Hausdorff measure; see Remark 4) we obtain both an optimal packing or covering [11], and the exact value of the corresponding measure. Our method requires a particular form of the separation condition, the SSC, in order to make the computation of the metric measures feasible (see Section 4 for a discussion of the computability of metric measures satisfying the OSC).

Metric measures suitable for studying the size of sets of Lebesgue measure zero in $\mathbb{R}^n$, such as the Hausdorff, packing, and centered Hausdorff measure (H^s, P^s, and C^s, respectively) have been studied intensively in recent years. However, the challenging problem of finding systematic methods for computing the values of these measures for a general fractal set remains open. Much effort has been made in this direction, and exact values and bounds for the measures of some fractal sets are known already (see [1–14,16,17,20,24], and the references therein).

In this direction, the algorithms presented in [13] and [14] can be seen as the first steps towards the systematic computation of the centered Hausdorff and packing measures of a self-similar set. These algorithms yield estimates of the corresponding measures for a wide class of self-similar sets, taking as input the list of contracting similitudes associated with the given set. It is important to note that in some cases, such as for the class of Sierpinski gaskets with dimension less than or equal to one, the packing measure algorithm has been useful not only for estimating the value of P^s on each particular set in the class, but also for finding a formula for the packing measure of an arbitrary member of the class. As shown in [14, Theorem 2], the information provided by the algorithm can then be used to prove the formula. However, in many other cases, such as for some plane self-similar sets of dimension greater than one, the absence of the corresponding formula means that it is desirable to know the accuracy of the numerical results obtained from the algorithms. In [13] the centered Hausdorff measure algorithm was implemented for some sets whose centered Hausdorff measures were available in the literature and some other sets whose centered Hausdorff measures were still unknown. It is remarkable that, in the first case, the optimal values were attained at early iterations and, in the second case, the algorithm yielded conjectural values that could be proved with the methods developed in [14]. However, the rate of convergence of neither algorithm was known. This is the problem we solve in this paper and the content of the next two main theorems.

Theorem 1. *Suppose that the system $\Psi = \{f_1, \dots, f_m\}$ satisfies the SSC. Then, for every $k \in \mathbb{N}^+$ such that $c - 3Rr_{\max}^k - 2Rr_{\max}^{k+1} > 0$ and every $\tilde{M}_k$ as in (18), there holds*

$$|P^s(E) - \tilde{M}_k| \le \varepsilon_k, \tag{4}$$

where

$$\varepsilon_k := \frac{s2^{s+1}RQ}{r_{\min}^{sq_k}} r_{\max}^k,$$

$s = \dim_H(E)$, $q_k \in \mathbb{N}^+$ *is such that*

$$Rr_{\max}^{q_k} \le c - R2r_{\max}^{k+1} - 2Rr_{\max}^k < Rr_{\max}^{q_k-1},$$

and

$$Q := \begin{cases} \left(\frac{c}{r_{\min}}\right)^{s-1} & \text{if } s \ge 1, \\ \left(c - 2Rr_{\max}^k - 2Rr_{\max}^{k+1}\right)^{s-1} & \text{if } s < 1. \end{cases} \tag{5}$$

Theorem 2. *Suppose that the system $\Psi = \{f_1, \dots, f_m\}$ satisfies the SSC. Then, for every $k \in \mathbb{N}^+$ and every $\tilde{m}_k$ given by (39), there holds*

$$|C^s(E) - \tilde{m}_k| \le \epsilon_k, \tag{6}$$

where

$$\epsilon_k := \frac{s2^{s+1}RQ}{r_{\min}^{qs}} r_{\max}^k,$$

$s = \dim_H(E)$, $q \in \mathbb{N}^+$ *is such that* $Rr_{\max}^q \le c < Rr_{\max}^{q-1}$, *and*

$$Q := \begin{cases} R^{s-1} & \text{if } s \ge 1, \\ c^{s-1} & \text{if } s < 1. \end{cases} \tag{7}$$

Here, $\dim_H(E)$ and R stand for the *Hausdorff dimension* and the *diameter* of the self-similar set E.

As discussed in Sections 2.3 and 3.3, one of the most important features of Theorems 1 and 2 is that they provide sharp bounds for the exact values of the corresponding measures. Moreover, these bounds yield automatically estimates of the corresponding measures, accurate in some cases to as many as 14 decimal places. In the difficult case of self-similar sets having dimensions greater than one, for which less is known, we give examples with five decimal place accuracy. For instance, applying Theorem 1 to the family of Sierpinski gaskets $\{S_r\}$ with $\dim_H(S_r) = -\frac{\log 3}{\log r}$ (see (26) for a definition), yields $P^s(S_{0.37}) \simeq 3.8728$ (see Table 3) and $P^s(S_{0.42}) \simeq 3.62$. We also get $P^s(K_{\frac{4}{10}}) \simeq 5.27$, where $K_{\frac{4}{10}}$ is the plane Cantor set of dimension $-\frac{\log 4}{\log 0.4}$. To our knowledge none of these estimates were previously known.

However, the most important consequence of the combination of Theorems 1 and 2 with the algorithms given in [13] and [14] is that it automatically provides an approximation to the value of the measure of any self-similar set satisfying the SSC. We remark that the precision of the results depends on the size of the contraction ratios. Namely, the accuracy achieved improves as the contraction ratios decrease (see [13,14] and Section 4 below for a detailed discussion). In particular, the examples given in Sections 2.3 and 3.3 show that the algorithms accurately recover the known values of these measures for sets with dimension less than one. Moreover, the results presented in this article serve to rule out certain potential formulas for some classes of self-similar sets.

The paper is divided into two main sections, one devoted to the packing measure and the other to the centered Hausdorff measure. In each case, we first recall the relevant algorithm from Llorente and Morán [13] or Llorente and Morán [14], although in the case of the centered Hausdorff measure we give some improvements to the algorithm. This is done in Sections 2.1 and 3.1. Then, we prove Theorems 1 and 2 at the ends of Sections 2.2 and 3.2, respectively. It is remarkable that these proofs do not use the convergence of the corresponding algorithms, so the present note provides shorter alternative proofs of their convergence. Finally, Sections 2.3 and 3.3 are devoted to analyzing the results obtained by applying Theorems 1 and 2 to the examples given in [13] and [14]. These numerical experiments have a twofold purpose. On the one hand they illustrate the theoretical results, showing how the algorithms perform in practice. On the other hand they offer quite complete information, previously unavailable in the literature, on the exact values of the packing and centered Hausdorff measures of the self-similar sets in three of the most classic families of self-similar sets, namely, the central Cantor sets in the line, the Sierpinski gaskets, and the Cantor sets in the unit square. Finally, in Section 4 we discuss the computability of metric measures on self-similar sets in view of the results obtained in this paper.

Notational Convention 3. We denote the open and closed ball with center x and radius r by $B(x, d) = \{y \in \mathbb{R}^n : |x - y| < d\}$ and