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Chaos and periodicity in Vallis model for El Niño

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ABSTRACT

We investigate a two-dimensional parameter-space of a three-parameter, three-variable, continuous-time dynamical system, namely the Vallis model for El Niño phenomenon. We report on modifications in this parameter-space, as a function of the third parameter which is varied. More specifically we report on organization of chaos and periodicity, showing the existence of periodic structures embedded in a chaotic region, which are organized in period-adding sequences.

1. Introduction

The El Niño phenomenon consists in the occurrence of an anomalously warm of the water in the eastern equatorial Pacific Ocean, on the coast of Peru and Ecuador, this fact having a great impact on the global climate. Mathematical models exist for El Niño [1], from the most complicated to the simplest. One of these simple models is the continuous-time Vallis model [2,3], which consists of a set of three autonomous first-order nonlinear ordinary differential equations. Therefore, the Vallis model for El Niño phenomenon is given by

$$\begin{aligned} \dot{x} &= By - C(x+p), \\ \dot{y} &= -y + xz, \\ \dot{z} &= -z - xy + 1, \end{aligned} \tag{1}$$

where x, y, z represent dynamical variables, and B, C, p are parameters. It was derived by using a box of fluid characterized by different temperatures in the west and east parts, and a wind-driven current, to model an equatorial ocean. According to Ref. [3], x represents the current, generated in part by the temperature gradient, y represents the difference of temperatures of the east and west parts of the ocean, and z is the sum of these temperatures. Parameter B governs the strength of the air-sea interaction and the vertical temperature difference, C is the ratio of time scales of decay of sea surface temperature anomalies to a frictional time scale, and p represents the average effect of equatorial winds.

The classical Lorenz model [4]

$$\begin{aligned} \dot{X} &= \sigma(Y - X), \\ \dot{Y} &= rX - Y - XZ, \\ \dot{Z} &= XY - bZ, \end{aligned} \tag{2}$$

can be obtained from the Vallis model (1) by setting $p = 0$, $b = 1$, and with adequate linear relations between the variables of both systems [1], *i.e.*, the Lorenz model (2) with $b = 1$, can be obtained by a linear transformation on the variables of the Vallis model (1) with $p = 0$. This similarity between the two systems, justified by the fact that both describe convection, is evidenced in Fig. 1, which shows in (a) a global view of the (B, C) parameter-space of the Vallis model for $p = 0$, and in (b) a global view of the (r, σ) parameter-space of the Lorenz model for $b = 1$. Color in diagrams of Fig. 1 is associated to the value of the largest Lyapunov exponent (LLE), according to the scale on the right column. Therefore, regions painted in yellow to red (LLE$>$ 0) refer to parameters for which the trajectory followed by the system in the phase-space is chaotic, while black regions (LLE$=$ 0) refer to parameters for which the trajectory is periodic. Detail about how the diagrams of Fig. 1 were constructed, are given in the next section.

Some analytical and numerical investigation of El Niño Vallis model (1) have already been reported [3,5–8]. More specifically, in [3] it is reported the occurrence of chaos in this system for $B = 102$, $C = 3$, and $p = 0$. In [5] is investigated the localization problem of compact invariant sets of nonlinear time-varying systems with the differentiable right-side, and Vallis model (1) is considered as an example. By using averaging theory, sufficient conditions for the existence of periodic solutions, as well as their kind of stability, are provided in Ref. [6]. A computer-assisted proof for

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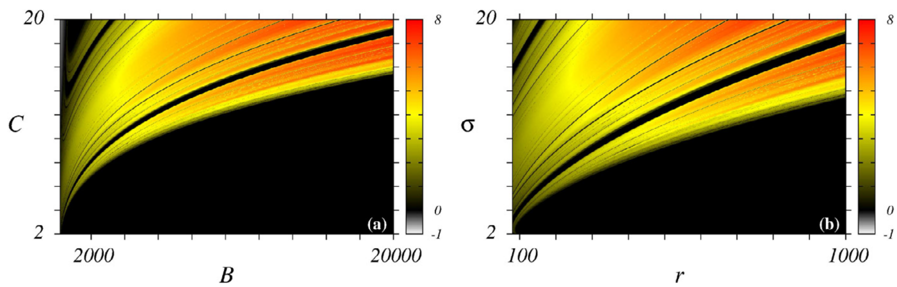

Fig. 1. (a) Global view of the (B, C) parameter-space of the Vallis model for $p = 0$. (b) Global view of the (r, σ) parameter-space of the Lorenz model for $b = 1$. In both diagrams are shown stability domains in the respective parameter-space, displaying chaotic regions in yellow to red color, and periodic regions in black. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

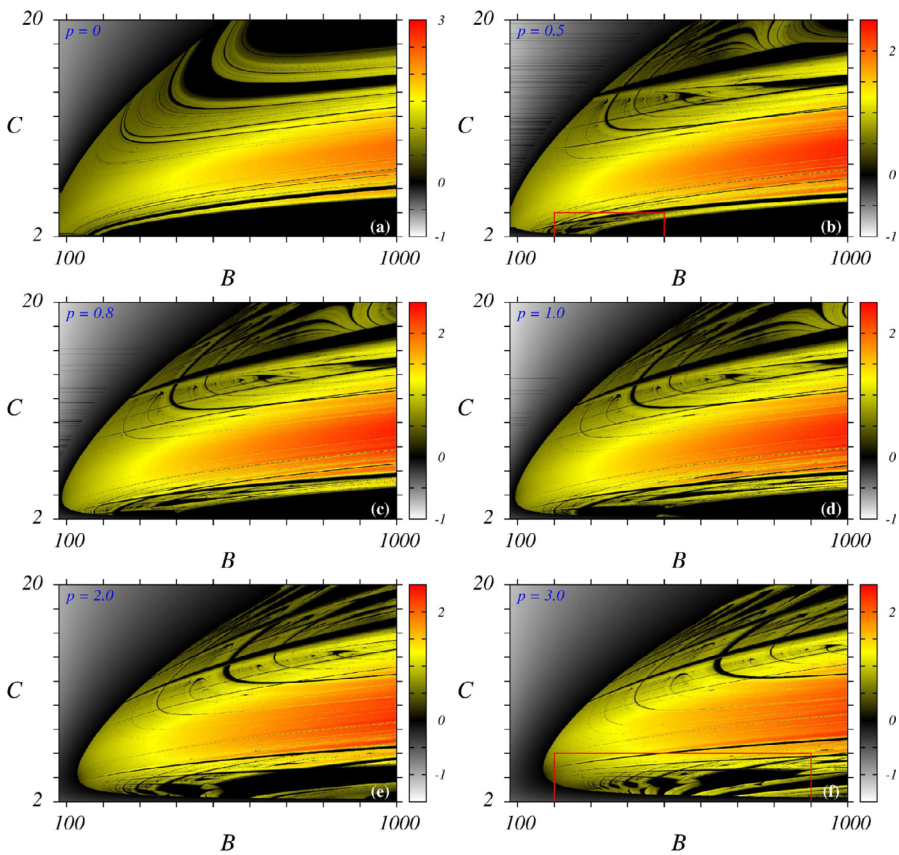

Fig. 2. Global view of the (B, C) parameter-space for system (1). As before in Fig. 1, are shown stability domains. (a) $p = 0$. (b) $p = 0.5$. (c). $p = 0.8$. (d) $p = 1.0$. (e) $p = 2.0$. (f) $p = 3.0$.

chaos in Vallis model (1) is reported by Garay and Indig [7] for $B = 102$, $C = 3$, and both for $p = 0$ and $p = 0.83$. Also, topological horseshoes were located by using Poincaré return maps. Reference [8] examined the model within the scope of fractional order derivatives. Three different definitions of derivative were considered to investigate the system under study, and the different results compared.

The aim of the present work is to investigate modifications in the (B, C) parameter-space of El Niño Vallis model (1), as a consequence of a discrete variation of the parameter p. We will show in the next section that, as p is increased from zero, typical periodic structures not present in the $p = 0$ parameter-space diagram, begin to appear embedded in a chaotic region. We will also show that some of these structures are organized in period-adding sequences, and that the higher the value of p, more pronounced is the phenomenon. As far as we know, investigations considering two-dimensional parameter-spaces of the El Niño Vallis model have not been reported in the literature.

2. The (*B*, *C*) parameter-space for El Niño Vallis model

In this section we present some (B, C) parameter-space diagrams for Vallis model (1), all of them obtained by considering the LLE value in each point of a $10^3 \times 10^3$ grid of equally spaced points, obtained by using the Wolf algorithm [9]. Regardless of the diagram considered, the range of parameters B and C is kept fixed. Different for each case is the value of the parameter p. Before the calculation of each LLE, system (1) was always integrated by using a fourth order Runge-Kutta algorithm, with a fixed time step size equal to 10^{-3}, being dropped the first 1×10^6 integration steps, regarded as a transient. For the computation of the average involved in the calculation of each one of the 1×10^6 LLE, were considered the subsequent 1×10^6 integration steps. Inte-