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A note on correlation and local dimensions☆

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ABSTRACT

Under very mild assumptions, we give formulas for the correlation and local dimensions of measures on the limit set of a Moran construction by means of the data used to construct the set.

1. Introduction

The correlation dimension was introduced in [14]. It is widely used in numerical investigations of dynamical systems. The properties of the correlation dimension and the L^q-spectrum has been studied for various types of attractors of iterated function systems; for example, see [3,12,13,15,18]. We continue this line of research and obtain our results Lemma 2.1 and Proposition 2.3 which complement the study initiated in [7].

The other important object in this paper is the local dimension of a measure. It has a close connection with the theory of Hausdorff and packing dimensions of a set. Therefore it is a classical problem to try to express the local dimension by means of the data used to construct the set; for example, see [1,2,9,10,17]. Our main result in this section is Theorem 3.6. Under a natural separation condition, the finite clustering property, it solves this problem completely.

2. Correlation dimension via general filtrations

Let (X, d) be a compact metric space and μ a locally finite Borel regular measure supported in X. Since the metric will always be clear from the content, we simply denote (X, d) by X. Recall that the support of a measure μ, denoted by $\operatorname{spt}(\mu)$, is the smallest closed subset of X with full μ-measure. For $s \geq 0$ and $x \in X$, define the *s-potential* of μ at the point x to be

$$\phi_s(x) = \int d(x,y)^{-s}\,\mathrm{d}\mu(y),$$

where $d(x, y)$ is the distance between two points x and y in X. Furthermore, define the *s-energy* of μ to be

$$I_s(\mu) = \int \phi_s(x)\,\mathrm{d}\mu(x) = \iint d(x,y)^{-s}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

For the basic properties of the s-energy, the reader is referred to the Mattila's book [11, §8]. The quantity

$$dim_{cor}(\mu) = \inf\{s : I_s(\mu) = \infty\} = \sup\{s : I_s(\mu) < \infty\}$$

is called the *correlation dimension* of the measure μ. Measure-theoretical properties of this dimension map are studied in [12].

We now recall the definition of the local dimension of measures. Let μ be a locally finite Borel regular measure on metric space X. The *lower* and *upper local dimensions* of the measure μ at a point $x \in X$ are defined respectively by

$$\underline{dim}_{loc}(\mu, x) = \liminf_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

$$\overline{dim}_{loc}(\mu, x) = \limsup_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Here $B(x, r)$ is the closed ball of radius $r > 0$ centered at $x \in X$. We also define the *lower Hausdorff dimension* of the measure μ by setting

$$\underline{dim}_H(\mu) = \operatorname{ess\,inf}_{x\sim\mu} \underline{dim}_{loc}(\mu, x).$$

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The correlation dimension of a measure μ is at most the lower Hausdorff dimension of the measure μ. We recall the proof of this simple fact in the following lemma.

Lemma 2.1. *If X is a compact metric space and μ is a finite Borel regular measure on X, then*

$$\underline{dim}_{loc}(\mu, x) = \inf\{s : \phi_s(x) = \infty\} = \sup\{s : \phi_s(x) < \infty\}$$

for all $x \in X$. Furthermore,

$$dim_{cor}(\mu) = \liminf_{r\downarrow 0} \frac{\log \int \mu(B(x,r))\,\mathrm{d}\mu(x)}{\log r} \le \underline{dim}_H(\mu).$$

Proof. Fix $x \in X$. If s is so that $\phi_s(x) < \infty$, then

$$r^{-s}\mu(B(x,r)) \le \int_{B(x,r)} d(x,y)^{-s}\,\mathrm{d}\mu(y) \le \phi_s(x) \quad \text{for all } r > 0. \tag{2.1}$$

It follows that $\underline{dim}_{\,loc}(\mu, x) \ge s$ and thus,

$$\underline{dim}_{loc}(\mu, x) \ge \inf\{s : \phi_s(x) = \infty\}.$$

To show that $dim_{cor}(\mu) \le \underline{dim}_{\,H}(\mu)$, fix $s > \underline{dim}_{\,H}(\mu)$. Notice that there exists a set A with $\mu(A) > 0$ such that $\underline{dim}_{\,loc}(\mu, x) < s$ for all $x \in A$. The above reasoning implies that $\phi_s(x) = \infty$ for all $x \in A$. Therefore $I_s(\mu) = \infty$ and the claim follows. Similarly, if $s < dim_{cor}(\mu)$, then, by integrating (2.1), we see that

$$r^{-s}\int \mu(B(x,r))\,\mathrm{d}\mu(x) \le I_s(\mu) < \infty \quad \text{for all } r > 0.$$

Therefore

$$\liminf_{r\downarrow 0} \frac{\log \int \mu(B(x,r))\,\mathrm{d}\mu(x)}{\log r} \ge dim_{cor}(\mu). \tag{2.2}$$

To show the remaining inequalities, fix $t < s < \underline{dim}_{\,loc}(\mu, x)$. Observe that now there exists $r_0 > 0$ such that $\mu(B(x, r)) < r^s$ for all $0 < r < r_0$. Thus

$$\begin{aligned}\phi_t(x) &= \int d(x,y)^{-t}\,\mathrm{d}\mu(y) = t\int_0^\infty r^{-t-1}\mu(B(x,r))\,\mathrm{d}r \\ &\le t\int_0^{r_0} r^{s-t-1}\,\mathrm{d}r + t\int_{r_0}^\infty r^{-t-1}\mu(B(x,r))\,\mathrm{d}r < \infty\end{aligned}$$

and $\inf\{s : \phi_s(x) = \infty\} \ge t$. The proof of the converse inequality of (2.2) is similar and thus omitted. □

Remark 2.2. (1) If there exist $A \subset X$ and $s, r_0, c > 0$ such that $\mu(B(x, r)) \le cr^s$ for all $0 < r < r_0$ and $x \in A$, then Lemma 2.1 implies that $dim_{cor}(\mu|_A) \ge s$. In particular, if μ is a finite measure, then for every $\varepsilon > 0$, there exists a compact set A with $\mu(X\backslash A) < \varepsilon$ such that $dim_{cor}(\mu|_A) \ge \underline{dim}_{\,H}(\mu)$. To see this, fix $\varepsilon > 0$ and let $\{s_i\}_{i\in\mathbb{N}}$ be a strictly increasing sequence converging to $\underline{dim}_{\,H}(\mu)$. Egorov's theorem implies that for every i, there are $r_i > 0$ and a compact set $A_i \subset X$ with $\mu(X \setminus A_i) < 2^{-i}\varepsilon$ such that $\mu(B(x,r)) < r^{s_i}$, for all $0 < r < r_i$ and $x \in A_i$. Defining $A = \bigcap_{i=1}^\infty A_i$, we have $\mu(X \setminus A) \le \sum_{i=1}^\infty \mu(X \setminus A_i) < \varepsilon$. Fix $N \in \mathbb{N}$ and let $B_N = \bigcap_{i=1}^N A_i$, then $\mu(B(x,r)) < r^{s_N}$, for all $0 < r < \min_{i\in\{1,\cdots,N\}} r_i$ and $x \in B_N \supset A$. This gives $dim_{cor}(\mu|_A) \ge s_N$ and, as N was arbitrary, finishes the proof.

(2) Let us consider the standard $\frac{1}{3}$-Cantor set and define μ_p to be the Bernoulli measure associated to the probability vector $(p, 1-p)$. It is well known that $dim_{cor}(\mu_p) = -\log_3(p^2 + (1-p)^2)$; for example, see Proposition 2.3. Recalling e.g. [4, Proposition 10.4], we see that $dim_{cor}(\mu_p) < \underline{dim}_{\,H}(\mu_p)$ for all $p \in (0, 1)\backslash\{1/2\}$.

If the metric space X is doubling, then we can define the correlation dimension via a discrete process. More precisely, we will see that the definition can be given in terms of general filtrations. These filtrations can be considered to be generalized dyadic cubes. This gives a way to calculate the correlation dimension in many Moran constructions; see Corollary 3.3.

Before stating the theorem, we recall the definitions of the doubling metric space and the general filtration. A metric space X is said to be *doubling*, if there is a doubling constant $N = N(X) \in \mathbb{N}$ such that any closed ball $B(x, r)$ with center $x \in X$ and radius $r > 0$ can be covered by N balls of radius $r/2$. A doubling metric space is always separable and the doubling property can be stated in several equivalent ways. For instance, a metric space X is doubling if and only if there are $0 < s, C < \infty$ such that if $\mathcal{B}$ is an r-packing of a closed ball $B(x, R)$ with $0 < r < R$, then the cardinality of $\mathcal{B}$ is at most $C(R/r)^s$. Here the *r-packing* $\mathcal{B}$ of a set A is a collection of disjoint closed balls having radius r. We write $\lambda B(x, r) = B(x, \lambda r)$ for $\lambda \in (0, \infty)$.

Now we give the definition of the general filtration. We assume that $(\delta_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$ are two decreasing sequences of positive real numbers satisfying

(F1) $\delta_n \le \gamma_n$ for all $n \in \mathbb{N}$,
(F2) $\lim_{n\to\infty} \gamma_n = 0$,
(F3) $\lim_{n\to\infty} \log\delta_n / \log\delta_{n+1} = 1$,
(F4) $\lim_{n\to\infty} \log\gamma_n / \log\delta_n = 1$.

For each $n \in \mathbb{N}$, let $\mathcal{Q}_n$ be a collection of disjoint Borel subsets of the doubling metric space X such that each $Q \in \mathcal{Q}_n$ contains a ball B_Q of radius δ_n and is contained in a ball B^Q of radius γ_n. Define

$$E = \bigcap_{n\in\mathbb{N}} \bigcup_{Q\in\mathcal{Q}_n} Q.$$

The collection $\{\mathcal{Q}_n\}_{n\in\mathbb{N}}$ is called the *general filtration* of E.

The classical dyadic cubes of the Euclidean space is an example of a general filtration. Such kind of nested constructions can also be defined on doubling metric spaces and these constructions also serve as examples. In Lemma 3.1, we show that certain Moran constructions are general filtrations. These constructions include, for example, all the self-conformal sets satisfying the strong separation condition.

Besides giving the desired discrete version of the definition, the following result states also that the correlation dimension is in fact the L^2-spectrum of the measure.

Proposition 2.3. *Let X be a compact doubling metric space. If $\{\mathcal{Q}_n\}_{n\in\mathbb{N}}$ is a general filtration of E and μ is a finite Borel regular measure on E, then*

$$dim_{cor}(\mu) = \liminf_{n\to\infty} \frac{\log \sum_{Q\in\mathcal{Q}_n} \mu(Q)^2}{\log \delta_n}.$$

Proof. Observe that for each $Q \in \mathcal{Q}_n$ we have $Q \subset B(x, 2\gamma_n)$ for all $x \in Q$. Therefore

$$\int \mu(B(x,2\gamma_n))\,\mathrm{d}\mu(x) = \sum_{Q\in\mathcal{Q}_n} \int_Q \mu(B(x,2\gamma_n))\,\mathrm{d}\mu(x) \ge \sum_{Q\in\mathcal{Q}_n} \mu(Q)^2$$

and it follows from Lemma 2.1 and (F4) that

$$\begin{aligned} dim_{cor}(\mu) &\le \liminf_{n\to\infty} \frac{\log \int \mu(B(x,2\gamma_n))\,\mathrm{d}\mu(x)}{\log 2\gamma_n} \\ &\le \liminf_{n\to\infty} \frac{\log \sum_{Q\in\mathcal{Q}_n} \mu(Q)^2}{\log \delta_n}. \end{aligned}$$

To show the other inequality, fix $r > 0$ and let $n \in \mathbb{N}$ be such that $\gamma_{n+1} \le r < \gamma_n$. Choose for each $Q \in \mathcal{Q}_n$ balls B_Q of radius δ_n and B^Q of radius γ_n so that $B_Q \subset Q \subset B^Q$. Now for each $Q \in \mathcal{Q}_n$ we have

$$Q \subset B(x, 2\gamma_n) \subset B_Q[4\gamma_n] \subset \bigcup_{Q'\in\mathcal{C}_Q} Q'$$