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On the analysis of pseudo-orbits of continuous chaotic nonlinear systems simulated using discretization schemes in a digital computer

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ABSTRACT

This paper reports the existence of more than one pseudo-orbit when simulating continuous nonlinear systems using a digital computer in a set-up different from the ones normally seen in the literature, that is, in a set-up where the step-size is not varied, the discretization scheme is kept the same as well as the initial conditions. Taking advantage of the roundoff error, a simple but effective method to determine a lower bound error and the critical time for the pseudo-orbits is used and the connection to the maximum (positive) Lyapunov exponent is established considering the bit resolution and the computational platform used for the simulations. To illustrate the effectiveness of the method and problems of using discretization schemes for simulating continuous nonlinear systems in a digital computer, the well-known Lorenz equations, the Rossler hyperchaos system, Mackey–Glass equation and the Sprott A system are used. The method can help the user of such schemes to keep track of the reliability of numerical simulations.

1. Introduction

Numerical computation plays a key role in analysing the solutions of nonlinear dynamical systems [8,11,28,48]. Numerical experiments [42] have been used since the seminal work of Lorenz [25] in order to understand complex nonlinear dynamical systems that exhibit chaotic behaviour. As a result, researchers have been identifying chaotic behaviour [41] in various systems by analysing the generated numerical solutions. These solutions are obtained using discretization schemes available in popular software and computers easily accessible to most researchers. However, as stated in [28], there are many published works in which the reliability of numerical results is not carefully verified. The same author states that “In the simple case of a dynamic discrete system (of Hénon map), there are doubts as to the nature of the computational results: long unstable pseudo-orbits or strange attractors?”.

At first sight the natural way to deal with the problem would be to borrow and adapt the results of earlier works on the linear case. However, this approach should be used with care since there are important differences between the linear case and the nonlinear case. One of the first papers to deal with the problem of roundoff errors for the linear case in a digital computer is [46], where the author studies the effects of roundoff in the floating-point realization of a general *linear* discrete filter governed by a *stable* difference equation. Further works on this direction with the objective of understanding and minimizing the roundoff error are [14,32,36,37,50] just to mention a few. Although limit cycles have been reported due to the roundoff [4,13,32], the basic idea was to design filter structures to reduce roundoff noise and coefficient sensitivity, avoid overflow oscillations and quantization limit cycles when magnitude truncation is employed. The problem of structural instabilities for the linear continuous case was studied in [34]. These results are useful to understand some consequences of the roundoff error but their extension to the nonlinear case is not completely obvious.

In the investigation of some of aforementioned problems in the context of nonlinear systems, Lorenz [26] coined the term “Computational Chaos” while studying the chaotic behaviour of difference equations used to approximate a continuous system represented by a set of differential equations as the step-size is increased. Further results on the same subject can be found in [8,18,27,53]. Lao [18], for instance, has introduced the concept of critical predictable time to provide a more precise description of computed chaotic solutions of nonlinear differential equations. The author has suggested that the computed solutions, using discretization schemes, can not lead to accurate long-term prediction of chaotic time-series beyond the critical predictable time. He has also pointed out that

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two digitally computed chaotic outputs generated by different discretization schemes differs after the critical predictable time even if the used initial conditions are exactly the same. In a similar way, Corless [5] points out four levels of abstraction when one is dealing with modelling: "the physical reality of the problem under study, the continuous mathematical model of that physical reality, the numerical discretization of that mathematical model, and the floating-point simulation of that discretization". Regarding specifically to nonlinear dynamics, Corless [5] is still more incisive when stating that "Results from one level may or may not transfer easily to another level, and in particular, even qualitative features may not be preserved in that transfer. Moreover, there is no inherent bias either way: in any change of level we can introduce or destroy chaos."

Having in mind not the occurrence of different solutions due to the increase of the step-size or to the use of different discretization schemes but due to the numerical solution itself, Nepomuceno [39] has shown that a simple sequence of iterations of a well-known nonlinear system can reach a steady state value that is not the theoretical one. In other words, the sequence did not converge to the theoretical value due to numerical issues. In the same work, a method to calculate the propagation of error in the computation of recursive function is presented. The investigation of propagation error is not a recent issue (See for instance [6,10,11]) but, in fact, there are many works based on deterministic or statistical tools that provide some confidence when simulating recursive functions. Analysing such functions, Nepomuceno [39] has calculated the propagation of the error based on the evaluation of the sequence of arithmetic functions and Taylor expansion. Although, the results seem reasonable, the application of such technique is not practical for recursive functions with many terms, such as nonlinear discrete models [3], or discretization schemes for obtaining the numerical solutions for nonlinear differential equations when the goal is to measure or, at least, to estimate the error.

In order to investigate the error in complex recursive functions, Nepomuceno and Martins [40] introduced an approach to evaluate a lower bound error based on the fact that although interval extensions [35] are mathematically equivalent, they may generate different computer simulation outcomes. The result of using multiple interval extensions is the introduction of a new concept, "multiple pseudo-orbits", that differs from the general view that a simulation generated by iterating nonlinear systems exhibits only one pseudo-orbit. To compare the generated pseudo-orbits to the true one, Hammel et al. [11] have shown that the latter exists near a pseudo-orbit using mathematical analysis. In a somehow similar context, this paper explores the lower bound error to the context of continuous nonlinear systems simulated using discretization schemes. The lower bound error, that is, an inferior limit for the error, has a direct consequence on the understanding of the solutions generated by nonlinear dynamical systems. By means of a very simple change in the equation to be simulated, that is, by applying a distributive property, two different pseudo-orbits are produced even when the initial conditions and step size are not varied.

To further emphasize the main point of the present work, consider a recent work in which it is addressed the issue of obtaining chaotic solutions in a finite interval of time using symbolic computation, extremely high-order Taylor expansion and a super computer [22]. The author states that " ... because Lorenz [26,27] further found that chaotic solutions are sensitive not only to initial conditions but also to numerical algorithms: different numerical algorithms with different time-steps may lead to completely different numerical results of chaos". The present paper goes beyond that, when it states that even when the initial conditions are exactly the same, the algorithm is not changed and the step time is kept unchanged, one can find multiple pseudo-orbits simply by changing the way the model is written. Instead of trying to give a full answer to question "which pseudo-orbit is more correct?", guidelines are provided to help the user of discretization schemes to analyse the variety of numerical solutions and to establish a relationship between the different pseudo-orbits and the maximum (positive) Lyapunov exponent when possible.

The rest of the paper is laid out as follows: In Section 2, two discretization schemes are briefly reviewed. The proposed method based on the lower bound error is presented in Section 3. To illustrate this approach, examples using the well-known Lorenz equations, the Rossler hyperchaos system, Mackey–Glass Equation and the Sprott A system are given in Section 4. Section 5 presents the conclusions.

2. Discretization schemes

Two discretization schemes are now briefly reviewed. The first scheme is the well-known Runge–Kutta of 4th order, usually referred to as RK4 [43]. Consider an initial value problem specified as follows:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{1}$$

where x is some state variable (or output signal).

With a step-size (or discretization step) $h > 0$ the RK4 can be expressed as

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{2}$$

where

$$\begin{aligned} K_1 &= f_n, \\ K_2 &= f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}K_1\right), \\ K_3 &= f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}K_2\right), \\ K_4 &= f(t_{n+1}, {}_n + hK_3). \end{aligned} \tag{3}$$

The second method investigated is the Monaco and Normand-Cyrot Discretization Scheme [33]. Let the dynamic system be

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{4}$$

where $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ are state variables, $\mathbf{f}(\cdot)$ are (differentiable) functions.

Here an alternative procedure for discretization of Eq. (4) as described in [2] is given. A discrete model of Eq. (4) can be written as

$$\mathbf{x}_{k+1} = g(\mathbf{x}_k, h), \tag{5}$$

where $\mathbf{x}_k \in \mathbb{R}^m$ are the discrete state variable at time $t = t_0 + kh$ and t_0 is the initial time.

In [19,30], the Monaco and Normand-Cyrot discretization scheme was obtained by the Lie exponential expansion of Eq. (4) as follows:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \sum_{n=1}^{\eta} \frac{h^n}{n!} L_f^n(\mathbf{x}_k), \tag{6}$$

where η is the expansion order. The Lie derivative is given by:

$$L_f(\mathbf{x}_k) = \sum_{j=1}^{m} f_j \frac{\partial \mathbf{x}}{\partial x_j}, \tag{7}$$

where f_j represents the jth component of the vector field. Higher order derivative orders can be calculated by:

$$L_f(\mathbf{x}_k) = L_f\left(L_f^{n-1}(\mathbf{x})_k\right). \tag{8}$$

All simulations are performed on an IBM PC-compatible machine using Matlab R2016a or Fortran. All routines used in this work are available upon request.