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Chaos in a low dimensional fractional order nonautonomous nonlinear oscillator

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ABSTRACT

We report the dynamics of a low dimensional fractional order forced *LCR* circuit using Chua's diode. The stability analysis is performed for each segment of the piecewise linear curve of Chua's diode and the conditions for the oscillation and double scroll chaos are obtained. The effect of fractional order on the bifurcation points are studied with the help of bifurcation diagrams. We consider both the derivatives of the systems current and voltage as fractional derivatives. When the order of the derivatives is decreased, the system exhibits interesting dynamical behavior. For instance, the value of the fractional order corresponding to the voltage is decreased, the chaotic regime in the system decreases. But in the case of current, the chaotic regime in the system increases initially and beyond a certain value of order, the chaotic regime decreases and extinguishes from the system. We found the lowest order for exhibiting chaos in the fractional order of the circuit as 2.1. For the first time, the experimental analogue of our proposed system is made by using the frequency domain approximation. The results are obtained from the experimental observations are compared with numerical simulations and found that they match closely with each other. The existence of chaos in the circuit is analyzed with the help of 0-1 test and power spectrum.

1. Introduction

Study on nonlinear dynamical systems and their complex behavior gained a great interest among the researchers for the past few decades [1–4]. Nevertheless, it is difficult to solve the dynamical systems by available analytical methods. Few analytical and several numerical methods are available depending on the specific situation. Especially, solving a fractional order dynamical systems are more complicated [5]. The equations representing fractional order dynamical system involve differentiation and integration of fractional order calculus. Having a history of 300 years, the fractional order calculus (FoC) was first referred by Leibniz and L'Hospital [6,7] and developed later on by many contributions. In recent years, the FoC has a variety of applications as it allows us to describe or model a real world dynamical systems more precisely than the classical integer order calculus. Studies reveal that real world dynamical systems are generally in fractional orders [8–11]. With its recent developments such as approximation method, the FoC has broadened its application area. To mention a few, physics, chemistry, engineering [12,13], electrical engineering [8,11,14], control systems [9,10,15], robotics [16], signal processing [17], chemical mixing [18], bio-engineering [19], electronic circuits [20,21] and mechanical oscillator [22].

Numerous studies are available on fractional order nonlinear systems [23–26]. Radwan et al. [27] studied, the stability of commensurate and incommensurate fractional order. A general procedure for studying the stability of a system with many fractional elements is also given. Ahmad et al. [28] investigated chaotic behavior in autonomous nonlinear models of fractional order. The linear transfer function approximations of the fractional integrator block are calculated for a set of fractional orders, based on frequency domain arguments, and the resulting equivalent models are studied. An electronic circuit model of tree shape to realize the fractional-order operator proposed by Chen Xiang Rong et al. [29]. Jia Rong et al. [30] reported fractional-order Lorenz system. They analyzed the system using the frequency-domain approximation method and the time-domain approximation method and reported its chaotic dynamics, when the order of the fractional-order system is varied. Chao Luo Rong et al. [31] investigated a fractional-order complex Lorenz system and its dynamical behaviors. The synchronization scheme in fractional-order complex Lorenz systems is also presented. Razminia Rong et al. [32] present an active control methodology for controlling the chaotic behavior of a

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fractional order version of Rössler system. Active synchronization of two chaotic fractional order Rössler systems is also investigated. Gejji Rong et al. [33], observed that inclusion of delay changes chaotic behavior in fractional-order Chen system. Ivo Petráŝ proposed a new fractional-order chaotic system based on the concept of Volta's system, has simple structure and can display a double-scroll attractor [34]. Recently, Wang Rong et al. [35] have shown the existence of chaos in fractional order Murali–Lakshamanan-Chua's (FoMLC) system by solving it numerically.

One way of solving the differential equation is through their analogue simulation of electronic circuits. Such study requires simple circuits that are available even in an undergraduate laboratory and off the shelf electronic circuit components. Analogue electronic simulation studies provide a quick scan of the entire parameter space in real time. Further, as they are purely RC based circuits the drawbacks of LC networks are done away with. Their simplicity and ease of implementation have led to not only validation of past theoretical predictions but also discovery of many new phenomena. The experimental study of chaos using nonlinear electronic circuits, is being an active topic of research, it could be extended to solve fractional order differential equations. Motivates by the above studies, in this manuscript we report the chaotic dynamics of a low dimensional electronic circuit in the frame of FoC. For the purpose, we consider the forced *LCR* circuit using piecewise linear nonlinearity [36–38]. The dynamics of our proposed circuit has been studied in a different genre like the analytical solution and cellular neural/nonlinear networks [39–41], etc.

Studying the dynamics of our system in the platform of fractional order, have become a possibility in nonlinear science such as in control, synchronization and secure communications. However, in the present work, a detailed analysis is made both numerically and experimentally in the laboratory. The stability condition for the existence of chaos, double scroll attractor and shifting of bifurcation points as a function of the order of the system are investigated in detail. We also study the system dynamics by considering both variable (current and voltage) as the fractional derivatives. Very interesting dynamical transitions are observed, when the system is made as an incommonsurate fractional order and when the system order is decreased from value one. For example, the order corresponding to the circuit voltage/current is decreased, the chaotic regime decreases (as a function of the amplitude of the external forcing). The system losses its chaotic (regime) behavior for a certain value of the fractional order. It is clearly evident from the bifurcation diagram. We define two bifurcation points as *first bifurcation point* and *last bifurcation point* which are corresponding to where the bifurcation starts and ends respectively. As a consequence of decreasing the fractional order, the nature of the chaotic regime, these two bifurcation points approaches each order and ultimately meets each other. Interestingly, we find that this decrement of chaotic behavior follows a polynomial of order four. Whereas in the case of derivative corresponding to the current is decreased, initially, the two bifurcation points move away from each other and afterward they move towards each other. Consequently, the chaotic regime in the system initially increases and then it decreases to zero. We find that the chaos exists in the system up to the order of 2.1. As a whole, the systems' order is 2.1 and to the best of our knowledge, this must be the lowest order for exhibiting chaos in a fractional second-order nonautonomous nonlinear systems.

In this work, for the first time an analogue circuit is constructed to our forced series *LCR* circuit and results obtained are matching with its numerical counter part. In order to confirm the chaotic behavior of the circuit, the experimental data was analyzed using 0-1 test, FFT spectrum, etc. The proposed system is found to exhibit all the dynamical behavior exhibited by the integer order original circuit. The experimental results obtained and are confirmed with numerical simulation of the normalized circuit equations. The paper is configured as follows. The theory of fractional order derivatives and integrals are discussed in Section 2. Section 3 deals with the implementation of our proposed circuit, stability analysis, period doubling bifurcation and confirmation test of chaos. The dynamics of the system as a function of fractional order is studied in Section 4. Section 5 concludes the work.

2. Fractional derivatives and integrals

The fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_t^q$. The continuous integro-differential operator is defined as

$$ {}_aD_t^q = \begin{cases} \dfrac{d^q}{dt^q} & q > 0, \\ 1 & q = 0, \\ \int_0^t (d\tau)^q & q < 0, \end{cases} $$

Many definitions are available for fractional derivatives and integrals. However, the Grünwald–Letnikov, the Riemann–Liouville and the Caputo definitions [10,18,42] are the frequently used ones. Out of the above Riemann–Liouville is found to be the simplest definition to use which is given by

$$ {}_0D_t^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau, \tag{1} $$

where $\Gamma(\,\cdot\,)$ is Euler's Gamma function and $(n-1) \le q < n$ [18]. When the fractional integral begin at t =0 and all the initial conditions are being zero, the Laplace transform of the Riemann–Liouville fractional derivative is

$$ L\left\{ \frac{d^q f(t)}{dt^q} \right\} = S^q L\{f(t)\}. \tag{2} $$

The fractional integral operator of order q is represented by the transfer function $F(s) = \frac{1}{S^q}$ in the frequency domain. The definitions of fractional integral do not allow direct implementation of the operator of complicated systems. It is necessary to develop the approximations for the fractional operators using the standard integer order operators to analyze the systems. Linear transfer function approximations of the fractional integrator for the orders from 0.1 to 0.9, based on frequency domain arguments, and the resulting equivalent models are given in [20,21]. We adopt this approximation to design our circuit.

3. Circuit implementation

Lot of studies are available in the literature about the forced series *LCR* circuit. The equation of motion of the oscillator is well known and it is explored in electrical circuits in the laboratory [39,43,44]. In the present section, we consider the original forced series *LCR* circuit [38] equation to convert it into fractional order. If we consider a fractional order model for each electrical element in the circuit realization, we can write a more general circuit equation. Here, the fractional order forced series *LCR* circuit equation becomes,

$$ \begin{aligned} C\frac{d^{q_1} v_c}{dt} &= i_L - h(v_c), \\ L\frac{d^{q_2} i_L}{dt} &= -i_L R - R_s i_L - v_C + F\sin(\Omega t). \end{aligned} \tag{3} $$

where,

$$ h(v_c) = \begin{cases} G_b v_c + G_a - G_b & v_c \ge B_P, \\ G_a v_c & |v_c| \le B_P, \\ G_b v_c - G_a + G_b & v_c \le -B_P, \end{cases} $$

Here q_1 and q_2 are the fractional orders of real electrical elements and the term $h(v_c)$ represents mathematical form of the piecewise