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Solution of Riemann problem for ideal polytropic dusty gas

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ABSTRACT

The Riemann problem for a quasilinear hyperbolic system of equations governing the one dimensional unsteady flow of an ideal polytropic gas with dust particles is solved analytically without any restriction on magnitude of the initial states. The elementary wave solutions of the Riemann problem, that is shock waves, rarefaction waves and contact discontinuities are derived explicitly and their properties are discussed, for a dusty gas. The existence and uniqueness of the solution for Riemann problem in dusty gas is discussed. Also the conditions leading to the existence of shock waves or simple waves for a 1-family and 3-family curves in the solution of the Riemann problem are discussed. It is observed that the presence of dust particles in an ideal polytropic gas leads to more complex expression as compared to the corresponding ideal case; however all the parallel results remain same. Also, the effect of variation of mass fraction of dust particles with fixed volume fraction (Z) and the ratio of specific heat of the solid particles and the specific heat of the gas at constant pressure on the variation of velocity and density across the shock wave, rarefaction wave and contact discontinuities are discussed.

1. Introduction

In recent years the solution of Riemann problem is widely used for the theoretical and numerical study of the system of conservation laws in ideal gasdynamics, non-ideal gasdynamics, magnetogasdynamics, reacting flows, shallow water theory etc. Riemann problem is an initial value problem for the one dimensional Euler equations supplemented by discontinuous initial data and its solution constitutes the basic building block for the construction of a solution to the general initial value problem Glimm [1]. The solution of the Riemann problem is composed of three waves, with always a contact discontinuity as the middle one while the other two are indifferently rarefaction or shock wave. If both external waves are rarefaction then it might occur to the formation of a vacuum region between two parts of the gas receding from each other. The Riemann problem for the ideal gas does not admit a solution in closed form. This has led several authors such as Godunov [2,3], Chorin [4], Smoller [5], Gottlieb and Groth [6], Quartapelle [7] and Toro [8] to develop iterative solution schemes to determine the different waves issuing from an initial discontinuity in the flow field variables. Two methods were first proposed by Godunov, one based on a fixed point scheme [2] and the other based on a higher order Newton's iterative scheme, with a tangent parabola instead of a straight line [3]. An experimental and numerical investigation of shock wave attenuation was studied by Berger et al. [9] in which they have focused on the dependency of the shock wave attenuation on a wide span of barrier geometries. In the case of Euler equations, the Riemann problem corresponds to the shock tube problem and for a detailed discussion of this, the reader is suggested to the book by Courant and Friedrich [10]. Lax [11] solved the Riemann problem for the case when the initial data consisting of constant initial states U_l^* and U_r^* are such that $\|U_l^* - U_r^*\|$ is sufficiently small; here U^* is the vector of conserved variable with U_l^* to the left of $x=0$ and U_r^* to the right of $x=0$ separated by a discontinuity at $x=0$. Exact solution of the Riemann problem has been studied by Godunov [2], Chorin [4] and Giacomazzo and Rezzolla [12]. The special solution of Euler equations in which one of the Riemann invariants remains constant throughout the flow field is called a simple wave. In simple wave solution, wave breaks and the solution has to be complemented by the introduction of shock wave. When the shock strength is small (i.e. weak shock) and even moderate, jumps in entropy and the Riemann invariants are surprisingly small, see Whitham [13]. Exact solution of weak shock waves in gasdynamics was studied by Singh et al. [14] for planar and nonplanar flows. The Riemann problem and elementary wave interactions of isentropic system in magnetogasdynamics were studied by Raja Shekhar and Sharma [15,16] and Liu et al. [17]. Mentrelli et al. [18] studied thoroughly the problem of interaction of waves originated from Riemann problem in an Euler fluid. The solution of Riemann problem in ideal and non-ideal isentropic magnetogasdynamics was studied by Sahadeb Kuila et al. see [19,20]. Further solution of Riemann

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problem in magnetogasdynamics was studied by Singh and Singh [21] in which they solved the Riemann problem analytically without any restriction on initial states.

Dusty gas is a mixture of gas and small solid particles, where the volume of solid particles should not be more than 5% of the total volume of the gas. The study of Riemann problem in dusty gas is of great interest due to its wide application in gas dynamics (See G. Rudinger [22]). Miura and Glass [23] studied theoretically the problem of propagation of shock wave through a dusty-air layer. Pai et al. [24] studied the similarity solution of strong shock wave propagation in a mixture of a gas and dust particles. Further Jena [25] studied the self similar solutions and converging shocks in a non-ideal gas with dust particles using Lie group transformation. Gupta et al. [26] have used a direct approach to analyze the solution of the Riemann problem for a dusty gas flow. The main motivation of the present work is to study the Riemann problem for the unsteady one-dimensional motion of ideal polytropic gas with dust particles without any restriction on the initial states. Here we have derived the explicit expression for shock waves, rarefaction waves and contact discontinuities in terms of fluid flow parameters (density, velocity, and pressure). The existence and uniqueness of the solution of Riemann problem in a dusty gas is discussed. And also those cases are discussed which gives information about the existence of shock waves or simple waves for a 1-family and for a 3-family of curves in a dusty gas. Also the effect of dust particles on the density and velocity profiles, for the case of shock wave and rarefaction wave, is also discussed.

2. Basic equations

Here we assume that the particles are spherical, of uniform size, incompressible and occupy less than 5% of the total volume, their specific heat is constant and the temperature is uniform within each particle, collisions between particles of different sizes are not considered. It is also assumed that the particles are uniformly distributed over the cross section of the duct, the size and distance between particles are small as compared with the cross sectional dimensions of the duct. The boundary layer effects and heat transfer with the duct walls are not considered, the particles are permanent i.e. no mass transfer takes place between the two phases. With the above assumptions, the governing equations describing a one dimensional planar flow of an ideal polytropic gas with dust particles may be written in the following form (see [22,23,25,27])

$$\rho_t + u\rho_x + \rho u_x = 0\,, \tag{1}$$

$$\rho(u_t + uu_x) + p_x = 0\,, \tag{2}$$

$$E_t + uE_x - \frac{p}{\rho^2}(\rho_t + u\rho_x) = 0\,, \tag{3}$$

where u is the velocity, ρ is the density, p is the pressure, t is the time and x is the spatial coordinate. The subscripts denote partial differentiation unless stated otherwise. The internal energy E per unit mass of the mixture is given by

$$E = \frac{(1-Z)p}{(\Gamma-1)\rho}. \tag{4}$$

Here, $Z=V_{sp}/V_g$ is the volume fraction and $k_p=m_{sp}/m_g$ is the mass fraction of the solid particles in the mixture where m_{sp} and V_{sp} are the total mass and volumetric extension of the solid particles respectively, V_g and m_g are the total volume and total mass of the mixture respectively, the Grüneisen coefficient $\Gamma=\gamma(1+\lambda\omega)/(1+\lambda\omega\gamma)$, with $\lambda=k_p/(1-k_p)$, $\omega=c_{sp}/c_p$, $\gamma=c_p/c_v$, where c_{sp} is the specific heat of the solid particles, c_p the specific heat of the gas at constant pressure and c_v the specific heat of the gas at constant volume. The entities Z and k_p are related via the expression $Z=\theta\rho$, where $\theta=k_p/\rho_{sp}$ with ρ_{sp} is the specific density of the solid particles.

For a polytropic dusty gas, the equation of state is

$$p = ke^{S/c_v}\left(\frac{\rho}{(1-\theta\rho)}\right)^{\Gamma}, \tag{5}$$

where k, c_v and γ are positive constants.

Using Eq. (4) in Eq. (3) we get

$$p_t + up_x + \frac{\Gamma p}{(1-\theta\rho)\rho}\rho u_x = 0. \tag{6}$$

Thus Eqs. (1), (2) and (6) can be written as

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{7}$$

$$u_t + uu_x + \frac{1}{\rho}p_x = 0, \tag{8}$$

$$p_t + up_x + \frac{\Gamma p}{(1-\theta\rho)\rho}\rho u_x = 0. \tag{9}$$

Eqs. (7)–(9) can be written in matrix form as

$$U_t + AU_x = 0, \tag{10}$$

where $U = \begin{bmatrix}\rho\\ u\\ p\end{bmatrix}$ and $A = \begin{bmatrix}u & \rho & 0\\ 0 & u & 1/\rho\\ 0 & \rho C^2 & u\end{bmatrix}$, with $C^2 = \frac{\Gamma p}{(1-\theta\rho)\rho}$.

The eigenvalues of the matrix A are

$$\lambda_1 = u - C,\ \lambda_2 = u \text{ and } \lambda_3 = u + C, \tag{11}$$

where C is the velocity of sound and is given as $C=(\Gamma p/((1-\theta\rho)\rho))^{1/2}$ and the corresponding eigenvectors are

$$K^1 = \begin{bmatrix}-\rho/C\\ 1\\ -\rho C\end{bmatrix},\quad K^2 = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} \text{ and } K^3 = \begin{bmatrix}\rho/C\\ 1\\ \rho C\end{bmatrix}. \tag{12}$$

Since all the eigenvalues of the matrix A are real and distinct, hence the system of Eq. (10) is strictly hyperbolic.

3. Riemann problem and the generalized Riemann invariants

Eq. (10) can be written in conserved form as

$$\frac{\partial U^*}{\partial t} + \frac{\partial F(U^*)}{\partial x} = 0, \tag{13}$$

where $U^*=(\rho,\ \rho u,\ \rho(u^2/2+E))^{tr}$, $F(U^*)=(\rho u,\ p+\rho u^2,\ \rho u(u^2/2+E)+pu)^{tr}$ with $E=(1-Z)p/((\Gamma-1)\rho)$.

Since, in a hyperbolic system each characteristic field is either linearly degenerate or genuinely non-linear according as $\nabla\lambda_i K^i=0$ and $\nabla\lambda_i K^i \neq 0$ respectively. Clearly, from Eq. (10) first and third characteristic fields are genuinely non-linear and hence will be either a shock or rarefaction, while second characteristic field is linearly degenerate and hence will be contact discontinuity. Here we consider only the case when wave is associated with the characteristic field which is either a shock or rarefaction.

The Riemann problem for the system of Eq. (13) is an initial-value problem with data of the form

$$U^*(x,0) = U_0^*(x) = \begin{cases} U_l^* & ,x<0 \\ U_r^* & ,x>0 \end{cases}, \tag{14}$$

where U_l^* and U_r^* are left and right constant state as defined in Eq. (13) and $x=0$ is point of discontinuity. The exact solution of the Riemann problem (13) and (14) has three waves, which is associated with the eigenvalues $\lambda_1=u-C$, $\lambda_2=u$ and $\lambda_3=u+C$ as shown