Chaos, Solitons and Fractals 95 (2017) 140–147

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

On the Lipschitz condition in the fractal calculus

Alireza K. Golmankhaneh[a,*], Cemil Tunc[b]

[a] Young Researchers and Elite Club, Urmia Branch, Islamic Azad University, Urmia, Iran
[b] Department of Mathematics, Faculty of Sciences, Yuzuncu Yil University, Van, 65080, Turkey

ARTICLE INFO

Article history:
Received 31 July 2016
Revised 29 November 2016
Accepted 6 December 2016

MSC:
26A33
28A80
34A08

Keywords:
Fractal calculus
Triadic Cantor set
Fractal Picard iteration
Fractal metric space
Fractal Cauchy sequence

ABSTRACT

In this paper, the existence and uniqueness theorems are proved for the linear and non-linear fractal differential equations. The fractal Lipschitz condition is given on the F^α-calculus which applies for the non-differentiable function in the sense of the standard calculus. More, the metric spaces associated with fractal sets and about functions with fractal supports are defined to build fractal Cauchy sequence. Furthermore, Picard iterative process in the F^α-calculus which have important role in the numerical and approximate solution of fractal differential equations is explored. We clarify the results using the illustrative examples.

1. Introduction

Fractional calculus involves derivatives and integrals with arbitrary orders applied in the problem of tautochrone, hydrogeology, viscoelasticity, non-conservative systems, heat conduction, anomalous diffusion models, chaotic systems, control theory, and etc [1–10]. Flow of the n-layers of viscous and heterogeneous liquid leads to the Newton's equation with the memory [11]. The box dimension of the fractal curves such as Weierstrass function has connection with order of local fractional differentiability. In the Levy distribution, the critical order of the characteristic function is related to Levy index [12]. A fractional system is suggested and the minimum effective dimension is derived. Meanwhile, the synchronization of two identical and nonidentical fractional systems utilizing the active control is explored [13]. In view of the causality principle and modeling of the sub-diffusive processes, the local fractional Fokker–Planck equation is derived using the Chapman–Kolmogorov condition [14,15]. Global existence and boundedness of solutions of a certain nonlinear integro-differential equation are presented [16,17]. Fractals are the geometrical shape with the fractal dimension which is bigger than their topological dimension. The techniques and methods to create analysis on the fractal sets and curves are studied by many researchers [18–20]. Recently, in the seminal paper F^α-calculus which involves the fractional local derivatives on the Cantor set and fractal curves have been established. The F^α-calculus is algorithmic which is useful is in the application [21–24]. The F^α-calculus which is a mathematical framework is applied to model the classical and quantum mechanics equations and Maxwell's equations on the fractal space-time [25–28]. Diffraction pattern for the case of the grating triadic Cantor sets is derived [29]. The Henstock–Kurzweil integration method is generalized for the unbounded and singular functions with the fractal support [30]. The non-local fractal derivatives are defined on the fractal Cantor set which is a mathematical model for the processes with memory effect on fractals [31].

The plan of the paper is as follows:

In Section 2, we have reviewed the F^α-calculus. In the third section we proceed with the study of existence and uniqueness theorems for linear fractal differential equations whose solutions are non-differentiable in the sense of standard calculus. In Section 4 we have indicated fractal Lipschitz conditions within F^α-calculus and corresponding definitions and theorems are given. The fractal Picard iteration method is extended to the case of the F^α-calculus in Section 5. Finally, in Section 6, we have conclusion of the paper.

* Corresponding author.
E-mail addresses: alirezakhalkili2002@yahoo.co.in, alirezakhalili2002@yahoo.co.in (A.K. Golmankhaneh), cemtunc@yahoo.com (C. Tunc).

http://dx.doi.org/10.1016/j.chaos.2016.12.001
0960-0779/

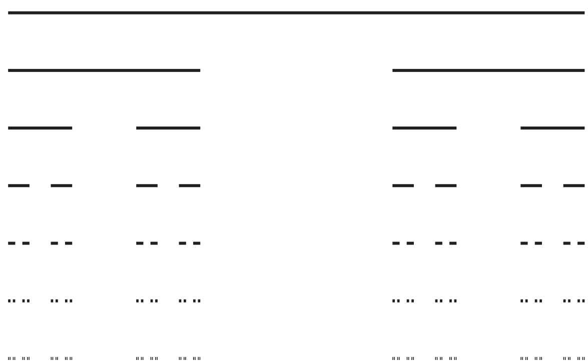

Fig. 1. We plot the iteration for the creation of the fractal Cantor set.

2. Basic tools in the fractal calculus

In this section, we summarize F^{α}-calculus without proofs [21].

2.1. The mass function and the integral staircase

If F is a fractal set, then it is the subset of $I=[a,b]$, a, $b\in\Re$ (Real-line). The flag function for F is shown by $\theta(F, I)$ and is defined [21],

$$\theta(F,I)=\begin{cases}1 & \text{if} \quad F\cap I\neq\emptyset\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

In Fig. 1 we show the iteration for the establishing of the fractal Cantor set with the dimension $\alpha=0.6$. The integral staircase function $S_F^{\alpha}(x)$ of order α for a fractal set F is defined in [21] by

$$S_F^{\alpha}(x)=\begin{cases}\gamma^{\alpha}(F,a_0,x) & \text{if} \quad x\geq a_0\\ -\gamma^{\alpha}(F,a_0,x) & \text{otherwise,}\end{cases} \tag{2}$$

where a_0 is an arbitrary real number.

A point x is a point of change of a function f if f is not constant over any open interval (a, d) involving x. The set $\mathbf{S}\mathrm{ch}f$ is called the points of change of f [21].

The γ-dimension of $F\cap[a, b]$ is

$$\begin{aligned}\dim_{\gamma}(F\cap[a,b]) &= \inf\{\alpha:\gamma^{\alpha}(F,a,b)=0\}\\ &= \sup\{\alpha:\gamma^{\alpha}(F,a,b)=\infty\}.\end{aligned} \tag{3}$$

In Fig. 2 we present the integral staircase function $S_F^{\alpha}(x)$. If $\mathbf{S}\mathrm{ch}(S_F^{\alpha})$ is a closed set and every point in it is limit point, so that $\mathbf{S}\mathrm{ch}(S_F^{\alpha})$ is called α-perfect. For the F-limit and the F- continuity definitions we refer the reader to [21].

2.2. F^{α}- differentiation

If F is an α-perfect set, then the F^{α}-derivative of f at x is [21]

$$D_F^{\alpha}f(x)=\begin{cases}F-\lim_{y\to x}\frac{f(y)-f(x)}{S_F^{\alpha}(y)-S_F^{\alpha}(x)}, & \text{if,} \quad x\in F,\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

if the limit exists.

2.3. Fundamental theorem of F^{α}-calculus

For a bounded function f on $F\cap[a, b]$, the fractal integral is defined as [21]

$$g(x)=\int_a^x f(y)d_F^{\alpha}y, \tag{5}$$

for all $x\in[a, b]$ and

$$D_F^{\alpha}g(x)=f(x)\chi_F(x). \tag{6}$$

Suppose f: $F\to R$ is F^{α}-differentiable function and $\mathbf{S}\mathrm{ch}(f)$ is contained in an α-perfect set F.

If h: $F\to R$ is F-continuous function, then we have

$$D_F^{\alpha}f(x)=h(x)\chi_F(x). \tag{7}$$

It follows that

$$\int_a^x h(x)d_F^{\alpha}x=f(b)-f(a). \tag{8}$$

The important formulas of the F^{α}-calculus [21]:

$$\int_0^y(S_F^{\alpha}(x))^n d_F^{\alpha}x=\frac{1}{n+1}(S_F^{\alpha}(y))^{n+1}, \tag{9}$$

$$D_F^{\alpha}(S_F^{\alpha}(x))^n=n(S_F^{\alpha}(x))^{n-1}\chi_F(x), \tag{10}$$

$$\int_a^b(af(x)+bg(x))d_F^{\alpha}x=a\int_a^b f(x)d_F^{\alpha}x+b\int_a^b g(x)d_F^{\alpha}x, \tag{11}$$

$$D_F^{\alpha}(a\ f(x)+b\ g(x))=a\ D_F^{\alpha}f(x)+b\ D_F^{\alpha}g(x), \tag{12}$$

$$D_F^{\alpha}c=0 \qquad \text{if} \quad c \quad \text{is} \quad \text{constant,} \tag{13}$$

$$D_F^{\alpha}(u(x)v(x))=(D_F^{\alpha}u(x))v(x)+u(x)(D_F^{\alpha}v(x)). \tag{14}$$

Here, our main results are stated by following definitions:

Definition 1. If F with distance $d:T_F^{\gamma}=F\times F\to\Re$, then

$$d_F^{\alpha}(S_F^{\alpha}(x),S_F^{\alpha}(y))=|S_F^{\alpha}(x)-S_F^{\alpha}(y)| \tag{15}$$

is called metric space associated with fractal sets and indicate by X_F^{α} (Fig. 3).

Definition 2. The sequences associated with fractal sets $\{S_{F_k}^{\alpha}(x), k=1,2,3,\ldots\}$ in a metric space X_F^{α} converges to $S_F^{\alpha}(x)$ if we have

$$\lim_{k\to\infty}d_F^{\alpha}(S_{F_k}^{\alpha}(x),S_F^{\alpha}(x))=0.$$

Definition 3. The sequences associated with fractal sets $\{S_{F_k}^{\alpha}(x), k=1,2,3,\ldots\}$ such that for every $\epsilon>0$ there exist some N such that $m,n>N$ as

$$d(S_{F_m}^{\alpha}(x),S_{F_n}^{\alpha}(x))<\epsilon,$$

is called a Cauchy sequence associated with fractal sets.

Definition 4. A metric space X_F^{α} is called complete if every Cauchy sequence associated with fractal sets converges to an element of X_F^{α}.

Definition 5. The uniform norm of a function f on a fractal set is defined by

$$||f||_F^{\alpha}=\sup_{x\in F}|f(S_F^{\alpha}(x))|. \tag{16}$$

Definition 6. The sequence of functions with the fractal support f_n converges uniformly to a function f with the fractal support if we have

$$F-\lim_{n\to\infty}||f_n-f||_F^{\alpha}=F-\lim_{n\to\infty}\sup_{x\in F}|f_n(S_F^{\alpha}(x))-f(S_F^{\alpha}(x))|=0. \tag{17}$$

Definition 7. Suppose $C_F^{\alpha}(X_F^{\alpha}\in[a,b],[c,d])$ which indicates the space of F-continuous functions f: $F\to[c, d]$. Namely, the F-continuous functions whose graphs lie inside the rectangle $\mathcal{R}=[a,b]\times[c,d]$. One builts the $C_F^{\alpha}(X_F^{\alpha}\in[a,b],[c,d])$ into a metric space by defining

$$d_F^{\alpha}(f,g)=||f-g||_F^{\alpha}. \tag{18}$$