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Complete characterization of algebraic traveling wave solutions for the Boussinesq, Klein–Gordon and Benjamin–Bona–Mahony equations

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ABSTRACT

In this paper, using a new method provided in [4] we characterize all algebraic traveling wave solutions of the fourth order Boussinesq equation, the nonlinear Klein–Gordon equation and a generalized Benjamin–Bona–Mahony equation.

1. Introduction and statement of the main results

Looking for traveling waves to nonlinear evolution equations has long been the major problem for mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields and thus they may give more insight into the physical aspects of the problems. Many methods for obtaining traveling wave solutions have been established [2,3,7,8,13,15] with more or less success. When the degree of the nonlinearity is high most of the methods fail or can only lead to a kind of special solution and the solution procedures become very complex and do not lead to an efficient way to compute them.

In this paper we first focus on the nonlinear Klein–Gordon equation [6,10–12] with quadratic nonlinearity terms

$$u_{tt} - u_{xx} + \alpha u - \beta u^2 = 0 \tag{1}$$

where $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$. The nonlinear Klein–Gordon equation appears in many types of nonlinearities and play a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory.

We also focus on the fourth order Boussinesq equation, which is a nonlinear partial differential equation of the form

$$u_{tt} - a^2 u_{xx} - b(u^2)_{xx} + u_{xxxx} = 0. \tag{2}$$

where $a, b \in \mathbb{R}$ with $b \neq 0$. This equation was introduced by Boussinesq to describe the propagation of long waves in shallow water. It arises in other physical applications such as nonlinear lattice waves, iron sound waves in a plasma and in vibrations in a nonlinear string (see, [5,6,10–12,16–19]).

Finally, we also focus on the modified Benjamin–Bona–Mahony equation

$$u_t + \alpha u_x + \beta (u^2)_x - \gamma u_{xxt} = 0 \tag{3}$$

where $\alpha, \gamma, \beta \in \mathbb{R}$ with $\beta \neq 0$. The Benjamin–Bona–Mahony equation was introduced by Benjamin, Bona and Mahony in [1] as an improvement of the KdV equation for modeling long surface gravity waves of small amplitude. It was introduced before by Peregrine in [9] in the study of undular bores.

There are various approaches for constructing traveling wave solutions, but these methods do not characterize completely when a given partial differential equation has a traveling wave solution in the sense that if they do not succeed in finding a traveling wave solution this does not mean that the system does not have a traveling wave solution. However, in [4] the authors gave a technique to prove the existence of traveling wave solutions for general nth order partial differential equations by showing that traveling wave solutions exist if and only if the associated n-dimensional first order ordinary differential equation has some invariant algebraic curve.

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More precisely, the traveling wave solutions of the partial differential Eqs. (1)–(3) are particular solutions, defined for all $s \in \mathbb{R}$, of the form $u = u(x,t) = U(x-ct)$. We will see that in the three cases $U(s)$ satisfies the differential equation

$$U'' = F(U, U'), \tag{4}$$

where F is a smooth map, $U(s)$ and the derivatives are taken with respect to s and the parameter c is called the *speed* of the traveling wave solution. Moreover, $U(s)$ satisfies the boundary conditions

$$\lim_{s\to-\infty} U(s) = A \quad \text{and} \quad \lim_{s\to\infty} U(s) = B, \tag{5}$$

where A and B are solutions, not necessarily different, of $F(u,0,0) = 0$.

We say that $u(x,t) = U(x-ct)$ is an algebraic traveling wave solution if $U(s)$ is a nonconstant function that satisfies (4) and (5) and there exists a polynomial $p \in \mathbb{R}[z,w]$ such that $p(U(s), U'(s)) = 0$.

We recall that for irreducible polynomials we have the following algebraic characterization of invariant algebraic curves: Given an irreducible polynomial of degree n, $g = g(x,y)$, we have that $g = 0$ is an invariant algebraic curve for the system $x' = P = P(x,y)$, $y' = Q = Q(x,y)$ for $P, Q \in \mathbb{C}[x,y]$, if there exists a polynomial $K = K(x,y)$ of degree at most $n-1$, called the cofactor of g such that

$$P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} = Kg.$$

The main result that we will use is the following theorem, see [4] for its proof.

Theorem 1. *The partial differential Eqs. (1)–(3) have an algebraic traveling wave solution with respect to c if and only if the first order differential system*

$$\begin{cases} y_1' = y_2, \\ y_2' = G_c(y_1, y_2), \end{cases} \tag{6}$$

where

$$G_c(y_1, y_2) = F(y_1, y_2)$$

has an invariant algebraic curve containing the critical points $(A, 0)$*and* $(B, 0)$ *and no other critical points between them.*

The main result is, with the techniques in [4], obtain all algebraic traveling wave solutions of Eqs. (1)–(3).

Theorem 2. *System (1) has an algebraic traveling wave solution if and only if* $c^2 \neq 1$, $\alpha \neq 0$. *The traveling wave solution is*

$$u(x,t) = \frac{3\alpha}{2\beta}\left(1 - \tanh^2\left(\frac{\sqrt{-a}}{2}\left(\sqrt{3}\kappa \pm (x-ct)\right)\right)\right), \quad \kappa \in \mathbb{R}$$

when $\alpha(c^2-1) < 0$ *being* $a = \alpha/(c^2-1)$, *and*

$$u(x,t) = \frac{\alpha}{2\beta}\left(-1 + 3\tanh^2\left(\frac{\sqrt{a}}{2}\left(\sqrt{3}\kappa b \pm (x-ct)\right)\right)\right), \quad \kappa \in \mathbb{R}$$

when $\alpha(c^2-1) > 0$, *being* $a = \alpha/(c^2-1)$ *and* $b = \beta/(c^2-1)$.

The proof of Theorem 2 is given in Section 3.

Theorem 3. *System (2) has the algebraic traveling wave solution*

$$u(x-ct) = \frac{c^2-a^2}{2b} + \frac{\bar{a}}{2b}\left(-2 + 3\tanh^2\left(\frac{\sqrt{\bar{a}}}{2}\left(\sqrt{3}\kappa_2 b \pm (x-ct)\right)\right)\right),$$

where $\bar{a} = \sqrt{(c^2-a^2)^2 - 4b\kappa_1}$, $\kappa_1, \kappa_2 \in \mathbb{R}$ *with* κ_1 *satisfying* $(c^2 - a^2)^2 - 4b\kappa_1 > 0$.

The proof of Theorem 3 is given in Section 4.

Theorem 4. *System (3) has an algebraic traveling wave solution if and only if* $\gamma c \neq 0$ *and the algebraic traveling wave solution is*

$$u(x-ct) = \frac{c-\alpha}{2\beta} + \frac{\bar{a}}{2b}\left(-2 + 3\tanh^2\left(\frac{\sqrt{\bar{a}}}{2}\left(\sqrt{3}\kappa_2 b \pm (x-ct)\right)\right)\right),$$

where $\bar{a} = \sqrt{(\alpha-c)^2 + 4\beta\kappa_1}$, $b = -\frac{\beta}{\gamma c}$, $\kappa_1, \kappa_2 \in \mathbb{R}$ *with* κ_1 *satisfying* $(\alpha-c)^2 + 4\beta\kappa_1 > 0$.

The proof of Theorem 4 is given in Section 5. For related results in the literature computing traveling wave solutions for variants of these systems but with other methods not being algebraic conclusive, see for instance [14] and the references therein.

2. Preliminary result

In this section we consider the general form that system (6) can have in each of the cases of Theorems 2–4. We study when it has an invariant curve passing through the singular points of the system and defining global solutions. Moreover we provide the form of these global solutions. The results in this section will be used in the proofs of Theorems 2–4.

Consider system

$$x' = y, \quad y' = -ax + bx^2, \tag{7}$$

where $a, b \in \mathbb{R}$ with $b \neq 0$. Note that system (7) has the algebraic invariant curve

$$g(x,y) = \frac{y^2}{2} + a\frac{x^2}{2} - b\frac{x^3}{3} = 0. \tag{8}$$

All solutions of (7) are on $g(x,y) = g$ for $g \in \mathbb{R}$ (note that $g(x, y)$ is a first integral of system (7)). So this curve is the only irreducible one. We search for constants g such that $g(x_0, 0) = g$ being $(x_0, 0)$ a singular point of system (7).

We first consider the case $a = 0$. In this case the zeroes of y' are on $x = 0$ and the unique possible value of g is $g = 0$. We thus have to solve

$$\frac{y^2}{2} - b\frac{x^3}{3} = 0$$

in y and we obtain

$$y = \pm\sqrt{\frac{2bx^3}{3}}.$$

Since $y(s) = x'(s)$, solving the linear differential equation

$$\frac{dx(s)}{ds} = \pm\sqrt{\frac{2bx^3}{3}}$$

the solutions are of the form

$$x(s) = \frac{12}{2bs^2 \pm 2\sqrt{6b}s\kappa + 3\kappa^2}, \quad \kappa \in \mathbb{R}.$$

Note that they are never global, so in this case there are no global solutions.

Assume now that $a \neq 0$. In this case the zeroes of y' are on $x = 0$ and $x = a/b$. So the unique possible values of g are $g = 0$ and $g = \frac{a^3}{6b^2}$. We will consider both cases separately.

In the first case when $g = 0$ we have that solving

$$\frac{y^2}{2} + a\frac{x^2}{2} - b\frac{x^3}{3} = 0$$

in y we obtain

$$y = \pm\sqrt{\frac{2bx^3}{3} - ax^2}.$$