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Review

Existence of limit cycles in a three level trophic chain with Lotka–Volterra and Holling type II functional responses

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ABSTRACT

In this paper we analyze a three level trophic chain model, considering a logistic growth for the lowest trophic level, a Lotka–Volterra and Holling type II functional responses for predators in the middle and in the cusp in the chain, respectively. The differential system is based on the Leslie–Gower scheme. We establish conditions on the parameters that guarantee the coexistence of populations in the habitat. We find that an Andronov–Hopf bifurcation takes place. The first Lyapunov coefficient is computed explicitly and we show the existence of a stable limit cycle. Numerically, we observe a strange attractor and there exist evidence of the model to exhibit chaotic dynamics.

1. Introduction

In the dynamic of the ecosystems, the formulation of many mathematical models with different interaction terms, including predation, competition, mutualism, like the trophic chains and other ecological models, have defined a new direction for the research in mathematical ecology (see Refs. [1,2,4,5]). For any trophic system, the most important state is the coexistence of the species in a cycle. According to studies of ecosystem models, in nature the only cycles that exist and remain are those that are stable from the ecological point of view, this means that they must be insensitive to perturbations of external elements [2]. Thus, an ecologically stable cycle must be isolated and mathematically must correspond to a limit cycle in the system of differential equations in the model of a trophic chain [6]. In fact, the existence of at least one stable limit cycle gives a succesfull explanation about communities in which it has been observed that populations oscillate periodically (see Refs. [3,7,8]), independent of external conditions. There are several papers dedicated to search limit cycles in different ecological models. For example, Sunaryo et al. [9], study the existence of limit cycles in a tritrophic chain with Lotka–Volterra functional response for the middle predator, and a Holling type-III functional response for the top predator. Upadhyay and Raw [10], examine the local and global stability of a trophic chain model with functional responses of Holling type IV for the interaction between species. The existence of simultaneous periodic orbits bifurcating from two zero-Hopf equilibria in a tritrophic chain model with functional responses of Holling type III was reported in [11]. Moreover, an analytical study of a triple Hopf bifurcation in a tritrophic food chain model with Holling type II functional responses was reported in [12]. For more details about food chain and Holling functional responses see the Appendix A.

In this paper, we analyze a three level trophic chain model where the lowest trophic level grows with a logistic equation in absence of the predators. We also consider that the functional response for the predator in the middle of the chain is Lotka–Volterra type and the functional response for the predator at the top of the chain is Holling type II. Additionally, the dynamic of the

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predator in the cusp of the chain is considered in the Leslie–Gower scheme, (see Ref. [13]). Denoting by $X(\tau)$, $Y(\tau)$ and $Z(\tau)$ the population densities of the three species, respectively, the model in this work is represented by the following system of nonlinear ordinary differential equations

$$\begin{aligned} \frac{dX}{d\tau} &= \rho X\left(1 - \frac{X}{k}\right) - a_1XY, \\ \frac{dY}{d\tau} &= ca_1XY - dY - \frac{a_2YZ}{Y + b_2}, \\ \frac{dZ}{d\tau} &= \sigma Z^2 - \frac{\beta Z^2}{Y + b_2}, \end{aligned} \tag{1.1}$$

where a_1, a_2, b_2, c, d, β, σ, ρ and k are positive parameters. The ecological meaning of each of these constants will be given in the next section.

The outline of this paper is as follows. In Section 2 we describe the three level trophic chain we shall consider. The analysis of the local stability at the equilibrium points is carried out in Section 3 and the analysis of Hopf bifurcation in Section 4. The Section 5 is devoted to the numerical analysis of our results, including the detection of the limit cycle with its phase portrait and its time series graphs, the evidence of a strange attractor and the degeneracy of the Hopf bifurcation. A concluding summary of our results comprises the last Section 6.

2. Ecological description model

The first equation of the model (1.1) describes the change in population density of the prey and is given by its own growth minus the rate at which preys are consumed by the first predator. The constant ρ is the intrinsic growth rate of the population of preys and k is the carrying capacity of the medium to sustain its population. We assume that, in the absence of the two predators, the prey population grows with a logistic equation limited by the carrying capacity k. The interaction term $-a_1XY$ says that the rate of predation upon the preys is proportional to the rate at which the predator Y and the prey X find each other (Lotka–Volterra scheme). The second equation of 1.1 describes the change in population density of predators Y, in which the first term represents its growth and the constant c is the conversion factor of a prey into a predator. The second term is the rate of natural mortality of predators in absence of preys and the third term is the rate decreasing the predator population Y by encounters with predator population Z (modeled with Holling functional response type II).The parameter a_2 is interpreted as the speed that the predator Z have an encounter with a predator Y per unit of density of predators Y. The ratio b_2/a_2 is the average time in the processing of a meal for the predator Z. Finally, the first term of the third equation expresses the growth in population density of predators Z, with σ as its growth rate. The second term is the rate of mortality in a Leslie–Gower scheme, as in the model studied in [13], where b_2 represents the residual loss of the predators due to the shortage of their favorite food. This last consideration, is based on the idea that the reduction of predator population, has a reciprocal relationship with the per capita availability of their favorite food, their ability charge is set by the environmental resources and is proportional to the abundance of their favorite food, here β is the ratio of intrinsic growth of the population divided by the conversion factor of the predator Y into a predator Z. In following section is carried out the analysis of the stability at the equilibrium points of the system (1.1) and its ecological interpretations.

3. Equilibrium points and its local stability

For ecological considerations, we restrict our analysis to the region

$$\Omega = \left\{(X, Y, Z) \in \mathbb{R}^3 : X \geq 0, Y \geq 0, Z \geq 0\right\}.$$

The interior of Ω correspond to the positive octant of $\mathbb{R}^3$. In the first step of the stability analysis, we find the equilibrium points of the food–chain system (1.1) in Ω. To do this, we solve the system of algebraic equations that result to equalling zero the right hand side of (1.1) and we obtain the equilibrium points

$$P_0^* = (0, 0, 0), \tag{3.1}$$

$$P_1^* = (k, 0, 0), \tag{3.2}$$

$$P_2^* = \left(\frac{d}{a_1c}, \frac{\rho B_1}{a_1^2ck}, 0\right), \tag{3.3}$$

$$P_3^* = \left(\frac{k(\rho\sigma - a_1B_2)}{\rho\sigma}, \frac{B_2}{\sigma}, \frac{\beta\left(\rho\sigma B_1 - a_1^2ckB_2\right)}{a_2\rho\sigma^2}\right). \tag{3.4}$$

where

$$B_1 = a_1ck - d, \quad \text{and} \quad B_2 = \beta - b_2\sigma.$$

The equilibrium P_1^* is consistent with the limit case in the sense that two predators are absent and the prey population density grows to reach, as a maximum, the carrying capacity k. To clarify the above claim, note that the solution to the equation of population growth of preys in the absence of predators is

$$X(\tau) = \frac{k}{1 - \left(1 - \frac{k}{X(0)}\right)e^{-\rho\tau}}$$

and when the time tends to infinite we obtain

$$\lim_{\tau\to\infty} X(\tau) = k.$$

We observe that the carrying capacity k is reached independently of the initial values. To guarantee that the equilibrium points P_2^* and P_3^* are in Ω, it is necessary to verify that the following conditions over the parameters are satisfied: $B_1 > 0$, $B_2 > 0$, $\rho\sigma > a_1B_2$ and $\rho\sigma B_1 > a_1^2ckB_2$. Ecologically, the equilibrium P_2^* opens the possibility that the predator Y and prey X can survive. It is important to observe that, ecologically, the equilibrium points of the form $(0, \tilde{Y}, 0)$, $(0, 0, \tilde{Z})$ and $(0, \tilde{Y}, \tilde{Z})$ are excluded, since the populations of predators die in the absence of their preys and the predators also become extinct if no other resource to modify their diet, this can be verified by simple inspection in equations. Finally, the equilibrium point P_3^* is the equilibrium where the three species coexist and for this reason is the most important for the analysis.

The next step in the stability analysis is the behaviour near of the equilibrium points P_2^* and P_3^*. For this purpose, we compute the linearization for the system (1.1) in each equilibrium point.

Dynamics in absence of the top–predator

In absence of the top–predator ($Z = 0$), the system (1.1) is reduced to the predator–prey system

$$\begin{aligned} \dot{X} &= \rho X\left(1 - \frac{X}{k}\right) - a_1XY, \\ \dot{Y} &= ca_1XY - dY. \end{aligned} \tag{3.5}$$

The model (3.5) has the equilibrium point $P_{E_2} = \left(\frac{d}{a_1c}, \frac{\rho B_1}{a_1^2ck}\right)$, which is located in the positive region of $\mathbb{R}^2$ when $B_1 > 0$. This