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# Periodic motions and chaos for a damped mobile piston system in a high pressure gas cylinder with *P* control

Donghua Wang, Jianzhe Huang*

*Department of Power and Energy Engineering, Harbin Engineering University, Harbin, Heilongjiang, 150001, China*

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ABSTRACT

In this paper, the complex motions for a moving piston in a closed gas cylinder will be analyzed using the discrete implicit maps method. The strong nonlinearity of such system will be observed due to the large quadratic and cubic stiffness. Period-1 motions which contain high order of harmonic components will be presented. The periodic motions will be discretized into multiple continuous mappings, and such mapping can be analyzed via Newton–Raphson iteration. The stability analysis will be given and the analytic conditions for the saddle-node and period-doubling bifurcation will be determined. From the semi-analytic solution route, the possible motions without considering the impact of the piston with the end wall of the cylinder will be obtained analytically. The scheme to reduce the vibration of the piston can be obtained through the parameter studies.



## 1. Introduction

The nonlinear systems have been extensively investigated with various analytic and numeric approaches over centuries. Since the motion for nonlinear system is difficult to predict, and it is highly sensitive to the initial conditions, one tried to seek an efficient approach to get the rigid solution when the trajectory converges to the equilibrium. Traditional methodologies, including perturbation method, harmonic balance method, and multiple scales method etc., can give approximate solution of periodic motions for nonlinear system with small parameters of nonlinear terms. However, those approaches are not capable of dealing with systems with strong nonlinearity which widely exists in the real applications.

At the end of the 18th century, Lagrange [1] studied periodic motions of three-body problem by perturbing the two-body problem using the method of averaging. Using the method of averaging, the average value of the amplitude of a slowly-varying oscillation over a cycle of the path can be calculated [2]. Poincare [3] developed the perturbation theory for periodic motions of celestial bodies. The solution is assumed to include linear parts and perturbed terms. A set of functions can be obtained for each order of perturbation parameters to determine the perturbed terms. Schwartz and Whitney [4] investigated the time- and space-periodic standing waves in deep water via a time-dependent conformal mapping method. The Stoke-type expansion was used by assuming the wave height to be small. Shawagfeh [5] gave an approximate solution for the nonlinear oscillator by expanding the nonlinear term as summation of s set of Adomian polynomials. Hill [6] solved the coupling equations for the amplitudes of the interfacial and surface wave fields in terms of Jacobian elliptic functions. In 2010, Ramlan et al. [7] considered a hardening stiffness in an energy harvesting device, and the effect of shifting the resonance frequency to increase the bandwidth of the system for such hardening spring has been studied. Liu et al. [8] proposed a nonlinear model to represent the series chemical reactions, and gave the analytic solution for such nonlinear system via symbolic computation. In 2012, Luo [9] developed a so-called generalized harmonic balance method to obtain the steady state solutions for the dynamical systems with strong nonlinearity. Luo and Huang [10] presented the bifurcation route of period-1 motions for a Duffing oscillator with cubic damping analytically using this method. Luo and Huang [12] then further investigated the mechanism for period-m motions to chaos in the Duffing oscillator. For the generalized harmonic balance method, the stability is determined by the eigenvalues of the Jacobian matrix for the periodic motion. For a complex motion which requires higher orders of approximation, its Jacobian matrix is the super-large sparseness matrix which costs great computation effort to get its eigenvalues and the accuracy cannot be guaranteed. To remove such drawback, Luo [12] developed a semi-analytical approach to determine periodic motions in nonlinear dynamical systems via discrete implicit maps. Luo and Guo [13] applied such technology to carry out the bifurcation tress of a Duffing oscillator. This semi-analytical method gave exact the same predictions as in

* Corresponding author.
*E-mail addresses:* hitwdh@163.com (D. Wang), jianzhe@hrbeu.edu.cn (J. Huang).

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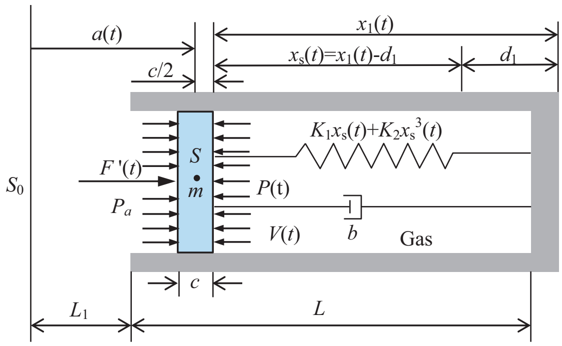


**Fig. 1.** Illustration of the mobile piston inside a cylinder with nonlinear spring and damper.

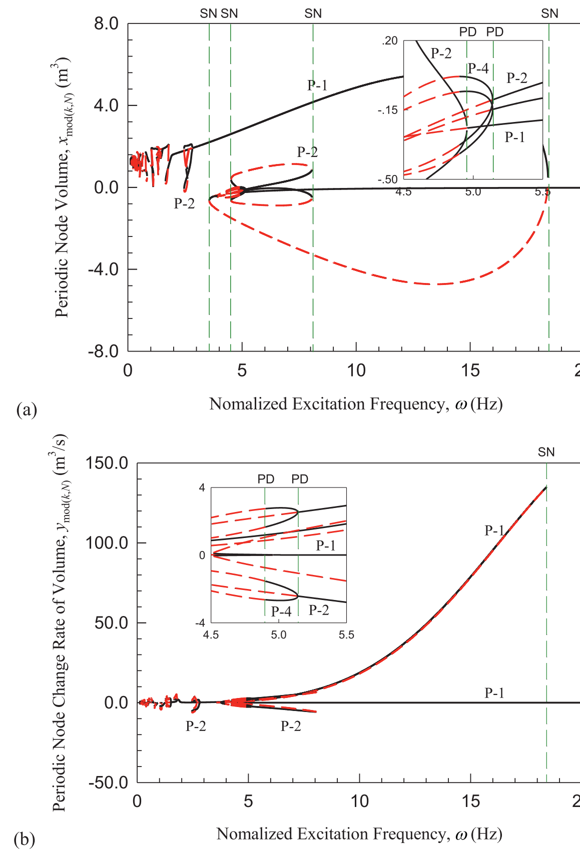


**Fig. 2.** Semi-analytic bifurcation route of period-1 to period-4 motions varying with normalized excitation frequency: (a) periodic node volume and (b) periodic node change rate of volume.

[10,11], and the dimension of the Jacobian matrix has been reduced to two for any kinds of periodic motions. Guo and Luo [14] continued to study the complex periodic motions and bifurcation trees of periodic motions to chaos in the Duffing oscillator with twin-well potential. For time-delayed nonlinear dynamical systems, the improved semi-analytical method was adopted to study the complex symmetric and asymmetric period-1 motions for a periodically forced, time-delayed, hardening Duffing oscillator in [15].

In this paper, a mobile piston model in a high pressure gas cylinder [16] will be studied. The thermal effect will be decoupled for the purpose of simplification. A $P$ controller will be considered in the controlled force in order to reduce the vibration of the piston. The bifurcation trees of period-1 motions to chaos in such nonlinear system will be presented using a semi-analytical method. The equation of motion for such mobile piston system will be discretized to get the discrete implicit maps. Based on the algebraic equations in mapping structures, the periodic motions for the mobile piston system without considering the boundary conditions can be predicted analytically. The analytic conditions of the period-doubling and saddle-node bifurcation for the periodic motion for such system will be discussed. Through the Fast Fourier Transform (FFT), nonlinear harmonic frequency-amplitude characteristic will be discussed. Numerical illustrations of periodic motions will be presented to interpret the mechanism of periodic motions to chaos.

## 2. System modeling and discretization

A mobile piston attached to a nonlinear spring and a damper is moving inside a closed cylinder which is illustrated in Fig. 1. The cylinder is filled with nitrogen, and the pressure inside the cylinder is extremely high. The piston has a mass $m$ and dimension $S \times c$, and it is excited with an external control force $F'(t)$. The atmospheric pressure is $P_a$, and the pressure inside the cylinder $P(t)$ is time varying. The linear and cubic spring stiffness is $K_1$ and $K_2$, respectively. The viscous damping is denoted by $b$.

$d_1$ is the initial length of the spring, and the displacement of the mobile piston from the standing surface $S_0$ is $a(t)$. Then the equation of motion for the piston can be derived as

$$m\frac{d^2a(t)}{dt^2} = F(t) - S\cdot P(t) + K_1[d - d_1 - a(t)] + K_2[d - d_1 - a(t)]^3 + b\frac{d[d - d_1 - a(t)]}{dt} \tag{1}$$

where

$$d = L + L_1 - c/2 \tag{2}$$

$$F(t) = F'(t) + S\cdot P_a \tag{3}$$

Let

$$\bar{d}_1 = d_1 - V_{sd}/S \tag{4}$$

Eq. (1) can be rewritten as

$$\frac{d^2V(t)}{dt^2} = \frac{S^2}{m}P(t) - \frac{S}{m}F(t) - \frac{K_1}{m}[V(t) - S\cdot\bar{d}_1] - \frac{K_2}{mS^2}[V(t) - S\cdot\bar{d}_1]^3 - \frac{b}{m}\frac{dV(t)}{dt} \tag{5}$$

with

$$V(t) = S[d - a(t)] - V_{sd} \tag{6}$$

where $V(t)$ is the gas volume and $V_{sd}$ is volume of the spring and damper.

To simplify the system, let $V_p(t) = V(t) - S\cdot\bar{d}_1$, then one can obtain

$$\frac{d^2V_p(t)}{dt^2} = \frac{S^2}{m}P(t) - \frac{S}{m}F(t) - \frac{K_1}{m}V_p(t) - \frac{K_2}{mS^2}V_p^3(t) - \frac{b}{m}\frac{dV_p(t)}{dt} \tag{7}$$

Consider the pressure in the closed cylinder is high, and the force upon the piston cannot be balanced unless the spring and damper also have prohibitive coefficient. Therefore, the control law for the external force has to be chosen as

$$\frac{S}{m}F(t) = \frac{S^2}{m}P(t) - f_p\frac{b}{m}\frac{dV_p(t)}{dt} - \frac{K}{mS}V_p^2(t) + \frac{K_p}{m}V_p(t) + Q_0\cos\Omega t \tag{8}$$

where $K$ is the constant of the control force; $K_p$ represents the proportional gain to avoid the piston impacting with the cylinder end; $f_p$ is a variable factor such that the friction effect is considered when $f_p = 0$ and removed when $f_p = 1$; $Q_0$ and $\Omega$ are the amplitude and frequency of the periodic force, respectively.