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Irving–Mullineux oscillator via fractional derivatives with Mittag-Leffler kernel

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ABSTRACT

Recently, Abdon Atangana and Dumitru Baleanu suggested a novel fractional operator based in the Mittag-Leffler function with non-singular and nonlocal kernel. In this paper using the newly established fractional operator, an alternative representation of the Irving–Mullineux oscillator via Atangana–Baleanu fractional derivative in Liouville–Caputo sense is presented. Numerical simulations are obtained using an iterative scheme via Sumudu-Picard iterative method. The existence and uniqueness of the solutions are studied in detail using the fixed-point theorem and some properties of the inner product and the Hilbert space. Numerical simulations of the special solutions were done and new chaotic behaviors are obtained.

1. Introduction

Fractional Calculus (FC) represents complex physical phenomena more accurate and efficient. Many real world dynamical systems can be represented by fractional operators. Several works have recently proved that the physical systems with dissipation can be clearly modeled more accurately by using fractional representations [1–8]. The Irving–Mullineux nonlinear oscillator equation was proposed by Irving and Mullineux in [9] and the fractional counterpart in the Liouville–Caputo sense was proposed by Abbas in [10].

In many papers in the literature, some researchers say, the fractional derivative portrays the memory effect, which has not been proven in practice. Fractional operator based relaxation equation and the corresponding diffusion equation models (which govern many physical phenomena such as heat, mass, pollutants or liquid transport through porous media; colloid or proteins moving in biosystems or even in ecosystems) have been widely investigated in multiple disciplines including physics, mathematics, hydrology among many others [11–14]. The authors in [15] considering charging and discharging processes of different capacitors in electrical RC circuit. It was shown that, the measured experimental results could be exactly obtained within the FC approach for the fractional order $\alpha \approx 0.998$. In the framework of mechanical systems, a rotary double inverted pendulum system was studied in [16], here the fractional order control method showed superiority over a sliding mode control structure, settling time shortening, and vibration weakening. In [17], the authors obtained the viscous damping coefficient experimentally and theoretically for the spring-mass viscodamper system using the FC approach, different values of non-local damping are obtained, these values represented dissipative effects that correspond to the nonlinear situation of the physical process (realistic behaviors that are non-local in time).

Recently, Michele Caputo and Mauro Fabrizio presented a novel operator based on the exponential function with regular kernel [18–28], this derivative has therefore an advantage over the Liouville-Caputo derivative because the full effect of the memory can be portrayed, nevertheless, due to their properties, some researchers have concluded that this operator can be view as filter regulator due to that the integral associate is not a fractional operator [29]. To solve the problem, Atangana and Baleanu suggested two fractional derivatives based on the generalized Mittag-Leffler function, these derivatives with fractional order in Liouville–Caputo and Riemann–Liouville sense have non-singular and non-local kernel and preserve the benefits of the Riemann–Liouville, Liouville–Caputo and Caputo–Fabrizio operators [29–37].

The main purpose in this paper is to show an alternative representation of the Irving–Mullineux oscillator via Atangana–Baleanu fractional derivative in Liouville–Caputo sense. Numerical simulations are obtained using an numerical scheme via Sumudu-Picard iterative method.

The paper is organized as follows: in Section 2, we outline the fundamentals to use the fractional calculus. Alternative representations of the Irving–Mullineux oscillator are derived and presented in Section 3. Finally, in Section 4 we conclude this manuscript.

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2. Atangana–Baleanu fractional operators

The Atangana–Baleanu fractional derivative in Liouville–Caputo sense is defined as follows [29–37]

$$ {}_0^{ABC}\mathcal{D}_t^{\gamma} f(t) = \frac{B(\gamma)}{1-\gamma}\int_0^t \dot{f}(\theta) E_{\gamma}\left[-\gamma\frac{(t-\theta)^{\gamma}}{1-\gamma}\right]d\theta, \qquad (1) $$

where $\frac{d^{\gamma}}{dt^{\gamma}} = {}_0^{ABC}\mathcal{D}_t^{\gamma}$ is an Atangana–Baleanu fractional derivative in Liouville–Caputo sense with respect to t, E_{γ} is the Mittag-Leffler function and $B(\gamma)$ is a normalization function and has the same properties as in Liouville–Caputo and Caputo–Fabrizio case.

The Laplace transform of (1) is defined as follows [29–37]

$$ \begin{aligned} \mathscr{L}[{}_0^{ABC}\mathcal{D}_t^{\gamma} f(t)](s) &= \frac{B(\gamma)}{1-\gamma}\mathscr{L}\left[\int_0^t \dot{f}(\theta) E_{\gamma}\left[-\gamma\frac{(t-\theta)^{\gamma}}{1-\gamma}\right]d\theta\right] \\ &= \frac{B(\gamma)}{1-\gamma}\frac{s^{\gamma}\mathscr{L}[f(t)](s) - s^{\gamma-1}f(0)}{s^{\gamma}+\frac{\gamma}{1-\gamma}}. \end{aligned} \qquad (2) $$

The Sumudu transform of the Atangana–Baleanu fractional derivative in Liouville–Caputo sense of a function $f(t)$ is defined as

$$ ST\left\{{}_0^{ABC}\mathcal{D}_t^{\gamma} f(t)\right\} = \frac{B(\gamma)}{1-\gamma}\left(\gamma\Gamma(\gamma+1)E_{\gamma}\left(-\frac{1}{1-\gamma}u^{\gamma}\right)\right) \times [ST(f(t)) - f(0)]. \qquad (3) $$

The Atangana–Baleanu fractional integral of order γ of a function $f(t)$ is defined as

$$ {}_0^{AB}I_t^{\gamma} f(t) = \frac{1-\gamma}{B(\gamma)}f(t) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_0^t f(s)(t-s)^{\gamma-1}ds. \qquad (4) $$

3. Irving–Mullineux model

Starting from the fractional order oscillator model presented by Abbas in [10], we introduce the Atangana–Baleanu fractional derivative in Liouville–Caputo sense to obtain the following fractional order system

$$ \begin{aligned} {}_0\mathcal{D}_t x_1(t) &= x_2, \\ {}_0^{ABC}\mathcal{D}_t^{\gamma} x_2(t) &= -x_1 - \alpha(x_2 + x_2^{2/3}), \end{aligned} \qquad (5) $$

with small positive parameters.

Lemma 1. *For a linear autonomous fractional order system [38]*

$$ \mathcal{D}_t^{\gamma} X(t) = AX(t), \qquad X(0) = X_0, \qquad 0 < \gamma \le 1, \qquad (6) $$

where $X(t) = (x_1, x_2, \ldots, x_n)^T \in \Re^n$ and $A \in \Re^n$. The following conditions be satisfied

1. *If the eigenvalues satisfy $|arg(eig(A))| > \frac{\pi\gamma}{2}$, is asymptotically stable.*
2. *If the eigenvalues satisfy $|arg(eig(A))| = \frac{\pi\gamma}{2}$, is stable, $eig(A)$ stands for the eigenvalue of the matrix A.*

Considering the trivial solution (0, 0), the system (5) has only one equilibrium solution and we can write

$$ \mathcal{D}_t^{\gamma} X(t) = \mu X(t) + \beta(X(t)), \qquad \beta(X(t)) = (0, \alpha x_2^{2/3})^T, \qquad (7) $$

where the Jacobian matrix μ corresponding to the linear system (5) is given by

$$ \mu = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha \end{bmatrix}, \qquad (8) $$

the eigenvalues of the matrix μ are given by

$$ \lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2-4}}{2}, \qquad (9) $$

if α is positive, occur three possible cases, when $\alpha < 2$ we get complex eigenvalues and the argument of the eigenvalue for this first case is $arg(eig(\lambda)) = -\tan^{-1}\frac{\sqrt{4-\alpha^2}}{\alpha}$. Now consider when $\alpha > \sqrt{2}$, the $|arg(eig(\lambda))| \ge \pi/4$, which will be greater than $\gamma\pi/2$ only if $\alpha < 1/2$. In this case the equilibrium is asymptotically stable; when $\alpha = 2$ we get repeated negative eigenvalues, we have $arg(eig(\lambda)) = \pi > \gamma\pi/2$, the equilibrium is asymptotically stable and when $\alpha > 2$, we have $arg(eig(\lambda)) = \pi > \gamma\pi/2$ which ensure the asymptotic stability of the trivial equilibrium, we conclude that the instability may arise only for the case when $\gamma > 1/2$ and the $\alpha > \sqrt{2}$.

3.1. Special solution

The aim of this section is to provide a special solution using an iterative scheme for the system described by (5). Applying the Sumudu transform (3) on both sides of the system (5) yields

$$ \begin{aligned} & S[x_1(t)] - x_1(0) = S[x_2(t)], \\ & \frac{B(\gamma)\gamma\Gamma(\gamma+1)}{1-\gamma}E_{\gamma}\left(-\frac{1}{1-\gamma}u^{\gamma}\right)S[x_2(t)] - x_2(0) \\ & \quad = S\left[-x_1(t) - \alpha(x_2(t) + x_2(t)^{2/3})\right], \end{aligned} \qquad (10) $$

rearranging (10) the following is obtained

$$ \begin{aligned} & S[x_1(t)] = x_1(0) + S[[x_2(t)], \\ & S[x_2 t)] = x_2(0) + \frac{1-\gamma}{B(\gamma)\gamma\Gamma(\gamma+1)E_{\gamma}\left(-\frac{1}{1-\gamma}u^{\gamma}\right)} \\ & \quad \cdot S\left[-x_1(t) - \alpha(x_2(t) + x_2(t)^{2/3})\right], \end{aligned} \qquad (11) $$

applying the inverse Sumudu transform on both sides of (11) we let

$$ \begin{aligned} & x_1(t) = x_1(0) + S^{-1}\{x_2(t)\}, \\ & x_2(t) = x_2(0) + S^{-1}\left\{\frac{1-\gamma}{B(\gamma)\gamma\Gamma(\gamma+1)E_{\gamma}\left(-\frac{1}{1-\gamma}u^{\gamma}\right)} \right. \\ & \quad \left. \cdot S\left[-x_1(t) - \alpha(x_2(t) + x_2(t)^{2/3})\right]\right\}, \end{aligned} \qquad (12) $$

now we obtain the following recursive formula for (12)

$$ \begin{aligned} & x_{1(n+1)}(t) = x_{1(n)}(0) + S^{-1}\left\{x_{2(n)}(t)\right\}, \\ & x_{2(n+1)}(t) = x_{2(n)}(0) + S^{-1}\left\{\frac{1-\gamma}{B(\gamma)\gamma\Gamma(\gamma+1)E_{\gamma}\left(-\frac{1}{1-\gamma}u^{\gamma}\right)} \cdot \right. \\ & \quad \left. \cdot S\left[-x_1(t) - \alpha(x_2(t) + x_2(t)^{2/3})\right]\right\}, \end{aligned} \qquad (13) $$

the solution (13) is provided by

$$ \begin{aligned} x_1(t) &= \lim_{n\to\infty} x_{1(n)}(t), \\ x_2(t) &= \lim_{n\to\infty} x_{2(n)}(t). \end{aligned} \qquad (14) $$

3.2. Stability analysis of iteration method

Let $(X, |\cdot|)$ be a Banach space and H a self-map of X. Let $z_{n+1} = g(H, z_n)$ be particular recursive procedure. The following conditions must be satisfied for $z_{n+1} = Hz_n$.

1. The fixed point set of H has at least one element.
2. z_n converges to a point $P \in F(H)$.
3. $\lim_{n\to\infty} x_n(t) = P$.

Theorem 1. *Let $(X, |\cdot|)$ be a Banach space and H a self-map of X satisfying*

$$ \|H_x - H_z\| \le \eta\|X - H_x\| + \eta\|x - z\|, \qquad (15) $$