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Analyzing properties of Deng entropy in the theory of evidence

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ABSTRACT

The theory of Evidence, or Shafer-Dempster theory (DST), has been widely used in applications. The DST is based on the concept of a basic probability assignment. An important part of this theory is the quantification of the information-based-uncertainty that this function represents. A recent measure of uncertainty (or information) in this theory, called the Deng entropy, has appeared as an interesting alternative to the measures presented so far. This measure quantifies the both types of uncertainty found in DST, then it is considered as a total uncertainty measure (TU). It is shown that this measure does not verify some of the essential properties for a TU in DST such as monotonicity, additivity and subadditivity. Also, the definition of this new measure produces other debatable situations. These shortcomings call in question the utility of this measure in applications.

1. Introduction

The theory of Evidence [8,21], mostly known as Dempster–Shafer's theory (DST), was presented as an extension of the classical probability theory (PT). In the DST a new concept, called *basic probability assignment* (bpa), was introduced to generalize the one of the probability distribution in the PT.

In the 90s were presented many studies about the uncertainty based information that a bpa can represent. The majority of the measures presented had as starting point the Shannon's entropy [22]. In DST, were found more types of uncertainty than in PT, as it is logical because DST includes the PT. Yager [23] makes the distinction in DST between two types of uncertainty called: *discord* (*randomness* or *conflict*) and *non-specificity* respectively. Harmanec and Klir [11] presented a total uncertainty measure (TU) in DST, i.e. a measure that quantifies both types of uncertainty, that has been justified by an axiomatic approach (Klir and Wierman [18]). They also established, for such type of measures, a set of five desired properties that a TU must verify. Abellán and Masegosa [2] extended that set adding the important property of the monotonicity.

As far, the upper entropy measure is the only measure that verifies all the basic required properties exposed in Klir and Wierman [18] and Abellán and Masegosa [2].

Very recently, a new measure called *Deng entropy* [7] has been presented as an alternative in DST. This function considers that the degree of uncertainty is strongly related with the number of possible alternatives. But this new measure presents some shortcomings that make us to be cautious if we want to use it in applications. We will see that this measure does not verify the majority of the above mentioned properties and has other weakness motivated by its definition. It is true that some of those properties could be questionable but other ones are very important and essential for such measures. We remark that a measure of that type must take into account the decreasing or increasing in information or at least not express an erroneous situation. Also, when two non-interactive evidences can be joined, the total amount of information can be not be increased or decreased by an uncertainty measure. Finally, if an evidence on a finite set can be decomposed on more simple sets, then the total amount of information can not be increased.

The paper is organized as follows. Section 2 reviews briefly the Dempster-Shafer Theory (DST). Section 3 is devoted to show some of the most important measures of uncertainty presented in DST, and exposes the set of basic properties necessary for such measures. Section 4 studied the set of properties verified by the new measure, and shows some of the shortcoming found on that measure. Section 5 is dedicated to the conclusions and future works.

2. Dempster–Shafer theory of Evidence

Let X be a finite set considered as a set of possible situations, $|X| = n, \wp(X)$ the power set of X and x any element in X. Dempster–Shafer theory is based on the concept of basic probability assignment. A *basic probability assignment* (bpa), also called *mass assignment*, is a mapping m: $\wp(X) \to [0, 1]$, such that $m(\emptyset) = 0$ and $\sum_{A \subseteq X} m(A) = 1$. A set A such that $m(A) > 0$ is called a *focal element* of m.

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Let X, Y be finite sets. Considering the product space of possible situation $X \times Y$ and m a bpa on $X \times Y$. The marginal bpa on X, $m^{\downarrow X}$ (and similarly on Y, $m^{\downarrow Y}$), is defined in the following way:

$$m^{\downarrow X}(A) = \sum_{R|A=R_X} m(R), \quad \forall A \subseteq X \tag{1}$$

where R_X is the set projection of R on X.

There are two functions associated with each basic probability assignment, a belief function, Bel, and a plausibility function, Pl: $Bel(A) = \sum_{B \subseteq A} m(B), \quad Pl(A) = \sum_{A \cap B \neq \emptyset} m(B)$. They can be seen as the lower and upper probability of A, respectively.

We may note that belief and plausibility are interrelated for all $A \in \wp(X)$, by $Pl(A) = 1 - Bel(A^c)$, where A^c denotes the complement of A. Furthermore, $Bel(A) \leq Pl(A)$.

On each bpa on a finite set X, there exists a set of associated probability distributions p on X, of the following way:

$$K_m = \{p |\ Bel(A) \leq p(A),\ \forall A \in \wp(X)\} \tag{2}$$

We remark that $Bel(A) \leq p(A)$ is, in this case, equivalent to $Bel(A) \leq p(A) \leq Pl(A)$. K_m is a closed and convex set of probability distributions, also called *credal set* in the literature.

3. Measures of uncertainty in DST

The classical measure of entropy [22] on probability theory is defined by the following continuous function:

$$S(p) = -\sum_{x \in X} p(x) \log_2(p(x)), \tag{3}$$

where $p = (p(x))_{x \in X}$ is a probability distribution on X, $p(x)$ is the probability of value x and $\log_2$ is normally used to quantify the value in bits.[1] The value $S(p)$ quantifies the only type of uncertainty presented on probability theory and it verifies a large set of desirable properties [18,22].

In DST, Yager [23] makes the distinction between two types of uncertainty. One is associated with cases where the information is focused on sets with empty intersections and the other one is associated with cases where the information is focused on sets with cardinality greater than one. They are called *discord* (*randomness* or *conflict*); and *non-specificity* respectively. So far, both types of uncertainty have been considered with the same level of importance in DST.

The following function, introduced by Dubois and Prade [9], has its origin in the classical Hartley measure [10] on classical set theory, and on the extended Hartley measure on possibility theory (Higashi and Klir [13]). It represents a measure of non-specificity associated with a bpa and it is expressed as follows:

$$I(m) = \sum_{A \subseteq X} m(A) \log_2(|A|). \tag{4}$$

$I(m)$ attains its minimum, zero, when m is a probability distribution. The maximum, $\log_2(|X|)$, is obtained for a bpa, m, with $m(X) = 1$ and $m(A) = 0,\ \forall A \subset X$. It is showed in the literature that I verifies all the required properties for such a type of measure. It was extended on more general theories than DST in Abellán and Moral [3].

Many measures were introduced to quantify the discord degree that a bpa represents [18]. One of the most representative discord functions was introduced by Yager [23]:

$$E(m) = -\sum_{A \subseteq X} m(A) \log_2 Pl(A). \tag{5}$$

But this function does not verify in DST all the required properties.

We can see in the literature, about measures of uncertainty in DST, that when it is presented a new composed measure, i.e. a measure which quantifies both types of uncertainty (a TU measure), then both parts of uncertainty are considered with the same weight. We mean that the upper and the lower possible values for each part are the same. We can cite some examples of this type of composite measures with the same weight for both parts: Lamata and Moral [19], Klir and Ramer [17], Klir and Parvitz [16], Maeda and Ichihashi [20], Abellán, Klir and Moral [1]. On the other hand, we think that it could be discussed if we consider that the non-specificity part can suppose an important difference between the DST and the PT, where only the part of discord appears.

Harmanec and Klir [11,12] proposed the measure $S^*(m)$, equal to the maximum of the entropy (upper entropy) of the probability distributions verifying $Bel(A) \leq \Sigma_{x \in A} p(x) \leq Pl(A),\ \forall A \subseteq X$. This set of probability distributions is the credal set associated with a bpa m, that we have noted as K_m.

Harmanec and Klir [12] proposed S^* as a total uncertainty measure in DST, but they do not separate both parts. Abellán, Klir and Moral [1], have proposed upper entropy as an aggregate measure on more general theories than DST, separating coherently discord and non-specificity. These parts can be also obtained in DST in a similar way. In DST, one can consider

$$S^*(m) = S_*(m) + (S^* - S_*)(m), \tag{6}$$

where $S^*(m)$ represents maximum entropy and $S_*(m)$ represents minimum entropy on the credal set K_m associated to a bpa m, with $S_*(m)$ coherently quantifying the discord part and $(S^* - S_*)(m)$ its non-specificity part. That measure has been successfully used in applications (see Abellán and Moral [4]). An algorithm to obtain that value in DST, and on more general theories, can be found in Abellán and Moral [6].

Very recently, Deng [7] have presented a new uncertainty measure named *Deng entropy* that can be considered as a new composed measure, quantifying discord and non-specificity.

This function, called E_d can be defined as follow for a bpa m on a finite set X:

$$E_d(m) = -\sum_{A \subseteq X} m(A) \log_2 \frac{m(A)}{2^{|A|} - 1}. \tag{7}$$

It can be separated in two functions measuring the both types of uncertainty in DST:

$$E_d(m) = -\sum_{A \subseteq X} m(A) \log_2 m(A) + \sum_{A \subseteq X} m(A) \log_2(2^{|A|} - 1), \tag{8}$$

where the first term quantifies the part of discord and the second one the part of non-specificity of a bpa.

The E_d measure arises with the idea to give more importance to the increase in uncertainty produced when the number of alternatives increases, i.e. on the non-specificity part. It is not agreed with the standard bounds of values for such type of measures. It can be observed that the upper bound for the part of discord can be notably smaller than the one for the non-specificity part. This will be analyzed in the following sections.

The part of discord of E_d is a natural extension of the Shannon's entropy and was studied in the 90′s. It verifies a set of interesting properties but not all the necessary ones. In the literature, it has been considered that there is no discord in a bpa when all the focal sets share, at least, an element (see Abellán and Moral [5], Abellán and Masegosa [2]). The E measure of Yager is also agree with that consideration, and attains a value of 0 in that case. But, in that situation, the measure used in E_d to quantify the discord can express a positive value, that is not coherent with that concept of discord.

The part of non-specificity of E_d is some similar to the Hartley measure but it emphasizes in a very strong way on the number

[1] Indifferently, log and $\log_2$ are used in the literature for this aim.