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Persistence of chaos in coupled Lorenz systems

Mehmet Onur Fen

Basic Sciences Unit, TED University, 06420 Ankara, Turkey

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ABSTRACT

The dynamics of unidirectionally coupled chaotic Lorenz systems is investigated. It is revealed that chaos is present in the response system regardless of generalized synchronization. The presence of sensitivity is theoretically proved, and the auxiliary system approach and conditional Lyapunov exponents are utilized to demonstrate the absence of synchronization. Periodic motions embedded in the chaotic attractor of the response system is demonstrated by taking advantage of a period-doubling cascade of the drive. The obtained results may shed light on the global unpredictability of the weather dynamics and can be useful for investigations concerning coupled Lorenz lasers.

1. Introduction

Chaos theory, whose foundations were laid by Poincaré [1], has attracted a great deal of attention beginning with the studies of Lorenz [2,3]. A mathematical model consisting of a system of three ordinary differential equations were introduced by Lorenz [3] in order to investigate the dynamics of the atmosphere. This model is a simplification of the one derived by Saltzman [4] which originate from the Rayleigh-Bénard convection. The demonstration of sensitivity in the Lorenz system can be considered as a milestone in the theory of dynamical systems. Nowadays, this property is considered as the main ingredient of chaos [5].

A remarkable behavior of coupled chaotic systems is the synchronization [6–10]. This concept was studied for identical systems in [9] and was generalized to non-identical systems by Rulkov et al. [10]. Generalized synchronization (GS) is characterized by the existence of a transformation from the trajectories of the drive to the trajectories of the response. A necessary and sufficient condition concerning the asymptotic stability of the response system for the presence of GS was mentioned in [11], and some numerical techniques were developed in the papers [10,12] for its detection.

Even though coupled chaotic systems exhibiting GS have been widely investigated in the literature, the presence of chaos in the dynamics of the response system is still questionable in the absence of GS. The main goal of the present study is the verification of the persistence of chaos in unidirectionally coupled Lorenz systems even if they are not synchronized in the generalized sense. We rigorously prove that sensitivity is a permanent feature of the response system, and we numerically demonstrate the existence of unstable periodic orbits embedded in the chaotic attractor of the response benefiting from a period-doubling cascade [13] of the drive. Conditional Lyapunov exponents [9] and auxiliary system approach [12] are utilized to show the absence of GS. Our results reveal that the chaos of the drive system does not annihilate the chaos of the response, i.e., the response remains to be unpredictable under the applied perturbation.

The idea of using perturbations to generate chaos in systems of differential equations was initiated in the studies [14–16], and extension of chaos in coupled systems was considered in [17–20]. In particular, the paper [19] was concerned with the extension of sensitivity and periodic motions in unidirectionally coupled Lorenz systems in which the response is initially non-chaotic, i.e., it either admits an asymptotically stable equilibrium or an orbitally stable periodic orbit in the absence of driving. On the contrary, in this paper, we investigate the dynamics of coupled Lorenz systems in which the response system is chaotic in the absence of the driving.

Another issue that was considered in [19] is the global unpredictable behavior of the weather dynamics. We made an effort in [19] to answer the question *why the weather is unpredictable at each point of the Earth* on the basis of Lorenz systems. This subject was discussed by assuming that the whole atmosphere of the Earth is partitioned in a finite number of subregions such that in each of them the dynamics of the weather is governed by the Lorenz system with certain coefficients. It was further assumed that there are subregions for which the corresponding Lorenz systems admit chaos with the main ingredient as sensitivity, which means unpredictability of weather in the meteorological sense, and there are subregions in which the Lorenz systems are non-chaotic. It was demonstrated in [19] that if a subregion with a chaotic dynamics influences another one with a non-chaotic dynamics, then the

E-mail address: monur.fen@gmail.com

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latter also becomes unpredictable. However, there is still an important question if chaos is suppressed under the interaction of two subregions whose dynamics are both governed by chaotic Lorenz systems. The results of the present study show that this is not the case, and the interaction of two chaotic subregions leads to the persistence of unpredictability under certain conditions.

The rest of the paper is organized as follows. In Section 2, the model of coupled Lorenz systems is introduced. Section 3 is devoted to the theoretical discussion of the sensitivity feature in the response system. Section 4, on the other hand, is concerned with the numerical analyses of coupled Lorenz systems for the persistence of chaos as well as the absence of GS. The existence of unstable periodic motions embedded in the chaotic attractor of the response is demonstrated in Section 5. Some concluding remarks are given in Section 6, and finally, the proof of the main theorem concerning sensitivity is provided in the Appendix.

2. The model

Consider the following Lorenz system [3]

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 &= -x_1 x_3 + r x_1 - x_2 \\ \dot{x}_3 &= x_1 x_2 - b x_3, \end{aligned} \tag{2.1}$$

where σ, r, and b are constants.

System (2.1) has a rich dynamics such that for different values of the parameters σ, r and b, the system can exhibit stable periodic orbits, homoclinic explosions, period-doubling bifurcations, and chaotic attractors [21]. In the remaining parts of the paper, we suppose that the dynamics of (2.1) is chaotic, i.e., the system admits sensitivity and infinitely many unstable periodic motions embedded in the chaotic attractor. In this case, (2.1) possesses a compact invariant set $\Lambda \subset \mathbb{R}^3$.

Next, we take into account another Lorenz system,

$$\begin{aligned} \dot{u}_1 &= -\overline{\sigma} u_1 + \overline{\sigma} u_2 \\ \dot{u}_2 &= -u_1 u_3 + \overline{r} u_1 - u_2 \\ \dot{u}_3 &= u_1 u_2 - \overline{b} u_3, \end{aligned} \tag{2.2}$$

where the parameters $\overline{\sigma}$, $\overline{r}$ and $\overline{b}$ are such that system (2.2) is also chaotic. Systems (2.1) and (2.2) are, in general, non-identical, since the coefficients σ, r, b and $\overline{\sigma}$, $\overline{r}$, $\overline{b}$ can be different.

We perturb (2.2) with the solutions of (2.1) to set up the system

$$\begin{aligned} \dot{y}_1 &= -\overline{\sigma} y_1 + \overline{\sigma} y_2 + g_1(x(t)) \\ \dot{y}_2 &= -y_1 y_3 + \overline{r} y_1 - y_2 + g_2(x(t)) \\ \dot{y}_3 &= y_1 y_2 - \overline{b} y_3 + g_3(x(t)), \end{aligned} \tag{2.3}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))$ is a solution of (2.1) and $g(x) = (g_1(x), g_2(x), g_3(x))$ is a continuous function such that there exists a positive number L_g satisfying $\|g(x) - g(\overline{x})\| \geq L_g \|x - \overline{x}\|$ for all $x, \overline{x} \in \Lambda$. Here, $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^3$. It is worth noting that the coupled system (2.1)+(2.3) has a skew product structure. We refer to (2.1) and (2.3) as the drive and response systems, respectively.

In the next section, we will demonstrate the existence of sensitivity in the dynamics of the response system.

3. Sensitivity in the response system

Fix a point x_0 from the chaotic attractor of (2.1) and take a solution $x(t)$ with $x(0) = x_0$. Since we use the solution $x(t)$ as a perturbation in (2.3), we call it a *chaotic function*. Chaotic functions may be irregular as well as regular (periodic and unstable) [3,21]. We suppose that the response system (2.3) possesses a compact invariant set $\mathscr{U} \subset \mathbb{R}^3$ for each chaotic solution $x(t)$ of (2.1). The existence of such an invariant set can be shown, for example, using Lyapunov functions [19,22].

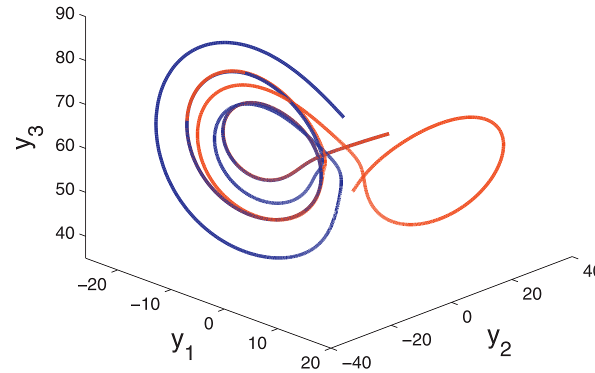

Fig. 1. Sensitivity in the response system (2.3). The simulation supports the result of Theorem 3.1 such that sensitivity is permanent in system (2.2) although it is driven by the solutions of (2.1).

One of the main ingredients of chaos is sensitivity [3,5,20]. Let us describe this feature for both the drive and response systems.

System (2.1) is called sensitive if there exist positive numbers ϵ_0 and Δ such that for an arbitrary positive number δ_0 and for each chaotic solution $x(t)$ of (2.1), there exist a chaotic solution $\overline{x}(t)$ of the same system and an interval $J \subset [0, \infty)$, with a length no less than Δ, such that $\|x(0) - \overline{x}(0)\| < \delta_0$ and $\|x(t) - \overline{x}(t)\| > \epsilon_0$ for all $t \in J$.

For a given solution $x(t)$ of (2.1), let us denote by $\phi_{x(t)}(t, y_0)$ the unique solution of (2.3) satisfying the condition $\phi_{x(t)}(0, y_0) = y_0$. We say that system (2.3) is sensitive if there exist positive numbers ϵ_1 and $\overline{\Delta}$ such that for an arbitrary positive number δ_1, each $y_0 \in \mathscr{U}$, and each chaotic solution $x(t)$ of (2.1), there exist $y_1 \in \mathscr{U}$, a chaotic solution $\overline{x}(t)$ of (2.1), and an interval $J^1 \subset [0, \infty)$, with a length no less than $\overline{\Delta}$, such that $\|y_0 - y_1\| < \delta_1$ and $\|\phi_{x(t)}(t, y_0) - \phi_{\overline{x}(t)}(t, y_1)\| > \epsilon_1$ for all $t \in J^1$.

The next theorem confirms that the sensitivity feature remains persistent for (2.2) when it is perturbed with the solutions of the drive system (2.1). This feature is true even if the systems (2.1) and (2.3) are not synchronized in the generalized sense.

Theorem 3.1. *The response system (2.3) is sensitive.*

The proof of Theorem 3.1 is provided in the Appendix. In the next section, we will demonstrate that the response system possesses chaotic motions regardless of the presence of GS.

4. Chaotic dynamics in the absence of generalized synchronization

Let us take into account the drive system (2.1) with the parameter values $\sigma = 10$, $r = 28$, $b = 8/3$ such that the system possesses a chaotic attractor [3,21]. Moreover, we set $\overline{\sigma} = 10$, $\overline{r} = 60$, $\overline{b} = 8/3$ and $g_1(x_1, x_2, x_3) = 2.95x_1 - 0.25 \sin x_1$, $g_2(x_1, x_2, x_3) = 3.06 \arctan x_2$, $g_3(x_1, x_2, x_3) = 3.12x_3 + 1.75e^{-x_3}$ in the response system (2.3). The unperturbed Lorenz system (2.2) is also chaotic with the aforementioned values of $\overline{\sigma}$, $\overline{r}$, and $\overline{b}$ [8,21].

In order to demonstrate the presence of sensitivity in the response system (2.3) numerically, we depict in Fig. 1 the projections of two initially nearby trajectories of the coupled system (2.1)+(2.3) on the $y_1 - y_2 - y_3$ space. In Fig. 1, the projection of the trajectory corresponding to the initial data $x_1(0) = -8.631$, $x_2(0) = -2.382$, $x_3(0) = 33.096$, $y_1(0) = 10.871$, $y_2(0) = -4.558$, $y_3(0) = 70.541$ is shown in blue, and the one corresponding to the initial data $x_1(0) = -8.615$, $x_2(0) = -2.464$, $x_3(0) = 33.067$,