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A gallery of chaotic systems with an infinite number of equilibrium points

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ABSTRACT

In this work, a systematic search for finding chaotic systems with infinite equilibria is described. As a result, we obtained a gallery of chaotic systems with various shapes of equilibrium points such as a line, two parallel lines, a piece–wise linear curve, a parabola, a hyperbola, or a circle. Interestingly, such novel systems exhibit "hidden attractors", which play vital roles in nonlinear theory and practical engineering issues.

1. Introduction

In last decades, chaos theory, systems with chaotic behavior, and chaos–based applications have been discovered [1–15]. Especially, three–dimensional (3D) chaotic systems have been received much attention in the literature [7]. There are different 3D chaotic systems, which have been studied such as Lorenz system [1], Rössler system [16], Arneodo system [17], Chen system [18], or Lü system [19]. There is a countable number of equilibrium points in these classical systems.

Recently, the presence of 3D chaotic systems with uncountable equilibria is reported [20–26]. The basin of attraction in such systems can intersect the equilibria in some portions. However there are usually uncountable equilibrium points which are located outside the basin of attraction. Therefore, we cannot identify the chaotic attractor by selecting an arbitrary initial condition in the neighborhood of the unstable equilibria [20]. In other words, knowledge about equilibria does not help in their localization or the attractor is hidden for computational purposes. As a result, from a computing point of view, such chaotic systems with infinite equilibria are considered as systems with "hidden attractors" [27–30]. Systems with "hidden attractor" have been of particular interest since "hidden attractor" can lead to unexpected phenomena [31–41]. There are two main research directions relating to chaotic systems with an infinite number of equilibrium points as illustrated in Fig. 1. On the one hand, motivated by the key work of Jafari and Sprott [20], authors focused on chaotic systems with line equilibrium. Five new chaotic flows with a line equilibrium and especially a complicated one with two infinite parallel lines of equilibrium points were discovered by Li and Sprott [21]. By using signum functions and absolute–value functions, chaotic systems with a line or two perpendicular lines of equilibrium points were introduced in [22]. On the other hand, after the discovery of a new class of chaotic systems with circular equilibrium [23], authors concentrated on chaotic systems with curve equilibrium. Kingni et al. presented a 3D chaotic autonmous system with a circular equilibrium and its fractional–order form [24]. Gotthans et al. reported another chaotic system with circular equilibrium in [23]. Moreover, authors also proposed a 3D system with a square equilibrium, which was constructed by modifying the system with circular equilibrium [25]. In addtion, a system exhibiting chaotic attractor with ellipse equilibrium, chaotic attractor with square–shaped equilibrium, and chaotic attractor with rectangle–shaped equilibrium was represented in [26].

It is worth noting that hyperchaotic systems with infinite equilibria have been investigated recently [42–46]. Zhou and Yang constructed a 4D hyperchaotic systems with infinitely many equilibria based on the Lü system [42]. Bistability was observed in a new hyperchaotic system with a line equilibrium [43]. Li et al. presented

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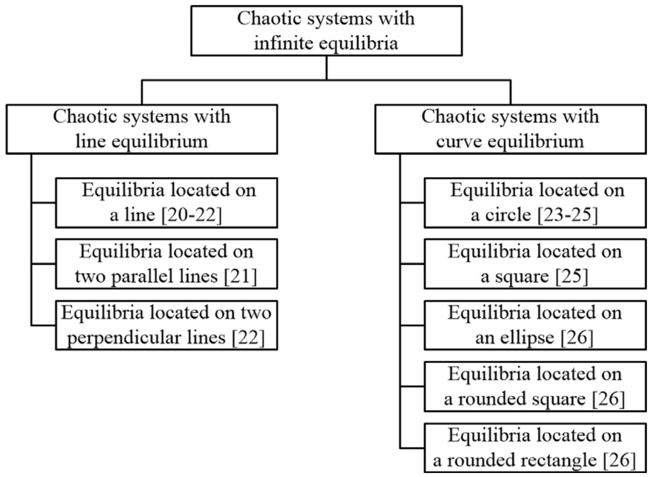

Fig. 1. Two main research directions for chaotic systems with infinite equilibria: discovering chaotic systems with line equilibrium and investigating chaotic systems with curve equilibrium.

hyperchaos in a small memristive neural network [44]. Hyperchaos and horseshoe in a 4D memristive system with a line of equilibria were studied in [45]. In addtion, Li et al. verified the existence of hyperchaos on the 4D memristive circuit [46].

The aim of our work is to contribute to the known list of chaotic systems with an infinite number of equilibria. New chaotic systems with different shapes of equilibria are found by using two general parametric forms.

2. Models for search routine

In the interesting work [20], Jafari and Sprott considered the conservative Sprott case A system:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + yz, \\ \dot{z} = 1 - y^2, \end{cases} \tag{1}$$

where x, y, z are state variables. Based on system (1), authors introduced a general parametric form to find chaotic flows with a line equilibrium:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = a_1 x + a_2 yz, \\ \dot{z} = a_2 x + a_3 y + a_4 xy + a_5 xz + a_6 yz + a_7 x^2 + a_8 y^2 + a_9 z^2, \end{cases} \tag{2}$$

where x, y, z are state variables and a_i are parameters. Dozens of chaotic flow with a line equilibrium were found and only nine simplest cases were reported [20].

Motivated by the published work of Jafari and Sprott [20], Gotthans and Petržela developed another predefined form to discover chaotic systems with circle equilibrium [23]:

$$\begin{cases} \dot{x} = az, \\ \dot{y} = z f_1(x, y, z), \\ \dot{z} = x^2 + y^2 - r + z f_2(x, y, z), \end{cases} \tag{3}$$

where r is radius of circular equilibrium, a is a free parameter while $f_1(x, y, z)$, $f_2(x, y, z)$ are two nonlinear functions. As a result, authors found a chaotic system with circular equilibrium given by

$$\begin{cases} \dot{x} = az, \\ \dot{y} = z(bx + cz^2), \\ \dot{z} = x^2 + y^2 - r + z(dx), \end{cases} \tag{4}$$

in which b, c, d are free parameters.

In this work, we present two new general parametric forms in order to investigate new chaotic systems with infinite equilibria. It is noting that absolute–value function is a potential nonlinear candidate to construct chaotic systems with hidden attractors [22,47,48]. Therefore, we include absolute terms into our general parametric forms. Based on system (2), we have constructed the first general parametric form as follows:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = a_1 x + a_2 yz, \\ \dot{z} = a_3|x| + a_4|y| + a_5 x + a_6 y + a_7 xy + a_8 xz + a_9 yz + a_{10} x^2 \\ \quad + a_{11} y^2, \end{cases} \tag{5}$$

where x, y, z are state variables and a_i are parameters. It is easy to see that there is a line of equilibrium points (0, 0, z) in system (5).

Based on system (3), we propose the second general form given by:

$$\begin{cases} \dot{x} = a_1 z, \\ \dot{y} = z f_1(x, y, z, |x|, |y|, |z|), \\ \dot{z} = f_2(x, y, |x|, |y|) + z f_3(x, y, z), \end{cases} \tag{6}$$

where

$$\begin{aligned} f_1(x, y, z, |x|, |y|, |z|) = {} & a_2|x| + a_3|y| + a_4|z| + a_5 x + a_6 y + a_7 z \\ & + a_8 xy + a_9 xz + a_{10} yz + a_{11} x^2 \\ & + a_{12} y^2 + a_{13} z^2 + a_{14}, \end{aligned} \tag{7}$$

$$\begin{aligned} f_3(x, y, z) = {} & a_{15} x + a_{16} y + a_{17} z + a_{18} xy + a_{19} xz + a_{20} yz \\ & + a_{21} x^2 + a_{22} y^2 + a_{23} z^2 + a_{24}, \end{aligned} \tag{8}$$

with a_i are parameters while $f_2(x, y, |x|, |y|)$ is a predefined nonlinear function. As can be seen, system (6) has an infinite number of equilibria which are located on the desired curve $f_2(x, y, |x|, |y|) = 0$ in the plane $z = 0$.

3. Obtained results

We have applied a reported search routine [49–54] into general models (5) and (6) for considering combinations of the coefficients and initial conditions. We have recorded the potential cases for which the largest Lyapunov exponent is greater than 0.001. For each found case, we have focused on its elegant feature [7]. It means that many coefficients as possible are set to zero with the others set to ± 1 or otherwise to a small integer or decimal fraction with the fewest possible digits [32,55]. This search routine has been introduced by Sprott and used widely in the literature [49–54]. It is worth noting that other search routines, for example a routine based on genetic programming algorithm [10], can be implemented to find new chaotic systems. However, we have only applied a known search routine and do not focus on novelty features of search routines in this work. The detail comparison with other search routines should be an interesting work.

We report eight elegant cases with the equilibria, values of parameters, Lyapunov exponents (LEs), Kaplan–Yorke dimensions (D_{KY}), and initial conditions ($x(0)$, $y(0)$, $z(0)$) (ICs) in Table 1. Lyapunov exponents in Table 1 are calculated by using the well–known Wolf's method [56]. Particularly such new systems contain a single absolute term only. In addition, phase portraits in the $y - z$ plane of such found systems with an infinite number of equilibrium points are shown in Fig. 2.

4. Discussions

In this section, we study further a simple example, system IE_1, with six terms, which has the following form:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + yz, \\ \dot{z} = a|x| - bxy - xz, \end{cases} \tag{9}$$

in which a and b are two positive parameters.