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Quantum continuous time random walk in nonlinear Schrödinger equation with disorder

A. Iomin*

Department of Physics, Technion, Haifa, 32000, Israel

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ABSTRACT

A quantum nonlinear Schrödinger equation in the presence of disorder is considered. The dynamics of an initially localized wave packet is studied and subdiffusion of the wave packet is obtained with a transport exponent 1/2. It is shown that this transport exponent has pure quantum nature. A probabilistic description of subdiffusion in the framework of a quantum continuous time random walk is suggested and a quantum master equation is obtained.

1. Introduction

In modern optical experiments on a quantum wave propagation in nonlinear media a suitable description can be suggested in the framework of the fractional kinetics based on fractional integro-differentiation. This concept of differentiation of non-integer orders arises from works of Leibniz, Liouville, Riemann, Grunwald, and Letnikov, see e.g., [1–3]. Its application is related to random processes with power law distributions. This corresponds to the absence of characteristic average values for processes exhibiting many scales [4,5].

A typical example of fractional dynamics in optics is realized in a competition between localization and nonlinearity that leads to anomalous transport [6–12] (See also a recent review [13]). This dynamics is described in the framework of the nonlinear Schrödinger equation (NLSE) in the presence of an external field $V = V(x),\ x \in (-\infty, +\infty)$. The wave spreading, described by the wave function, is governed by the nonlinear Schrödinger equation in the presence of disorder

$$i\partial_t \psi = -\partial_x^2 \psi + \beta|\psi|^2\psi + V\psi\,, \tag{1.1}$$

where β is a nonlinearity parameter. The variables are chosen in dimensionless units and the Planck constant is $\hbar = 1$. The random potential $V = V(x),\ x \in (-\infty, +\infty)$ leads to Anderson localization for the linear case ($\beta = 0$) [14,15]. The system is described by the exponentially localized Anderson modes (AM)s $\Psi_{\omega_k} \equiv \Psi_k(x)$, such that $[-\partial_x^2 + V(x)]\Psi_k(x) = \omega_k\Psi_k(x)$, where $\Psi_{\omega_k}(x)$ are real functions and the eigenspectrum ω_k is discrete and dense [15]. The problem in question is an evolution of an initially localized wave function $\psi(t=0) = \psi_0(x)$. It can be also a stationary state of the NLSE, which is localized with the same Lyapunov exponent [16,17].

This problem is relevant to experiments in nonlinear optics, for example disordered photonic lattices [18,19], where Anderson localization was found in the presence of nonlinear effects, as well as to experiments on Bose–Einstein condensates in disordered optical lattices [20–23]. A discrete analog of Eq. (1.1) is extensively studied numerically [6–9,12], and a subdiffusive spreading of the initially localized wave packet has been observed, such that $\langle x^2(t)\rangle = \int|\psi(t)|^2x^2dx \sim t^\alpha$, where a transport exponent α was found to be 2/5 [9] and 1/3 [8]. This difference has been explained in [24], where it has been shown that the former result is for the Markovian subdiffusion due to the range-dependent diffusion coefficient, while the latter one corresponds to non–Markovian fractional diffusion in a regime of percolation. In that case the dynamics of the wave packet has been approximated by the fractional Fokker–Planck equation (FFPE) due to the arguments of a so-called continuous time random walk (CTRW).

A subdiffusive spreading of the wave packet was also obtained analytically [10,24] in the limit of the large times asymptotic dynamics, where the transport exponent α has been found as well.

The concept of the CTRW was originally developed for mean first passage time in a random walk on a lattice with further application to a semiconductor electronic motion [25]. The mathematical apparatus of the fractional CTRW is well established for many applications in physics, see e.g., [4,5,26–28].

* Fax.: +972 4 829 5755.
E-mail address: iomin@physics.technion.ac.il

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In the present research we present the physical mechanism of subdiffusion of a quantum wave packet spreading and propose a quantum master equation based on the CTRW consideration, related to the quantum properties of the nonlinear interaction term in Eq. (1.1). To this end we consider a quantum counterpart of the NLSE (1.1) (quantum NLSE), when the wave functions $\psi(x)$ are considered as operators $\hat{\psi}(x)$ satisfying to the commutation rule $[\hat{\psi}(x), \hat{\psi}^{\dagger}(x')] = \delta(x - x')$. The paper consists of two parts. The first one is a quantum analysis which is based on mapping of the quantum system on the basis of the coherent states, and in the framework of the obtained equations we study four-mode decays to understand quantum continuous time walks and construct a generalized master equation as a quantum counterpart of the Liouville equation. The second part is a classical analysis of the c-number Liouville equation for the mean probability amplitude $\langle|\hat{\psi}(x,t)|^2\rangle$, where the transition elements in the Liouville operator are overlapping integrals of the AMs. Therefore, the further analysis is an application of the CTRW approach to the corresponding Liouville equation [29]. We also show that subdiffusion of the quantum wave packet with a transport exponent $\alpha = 1/2$ has a quantum nature.

2. Quantum NLSE

We concern with a quantum counterpart of the NLSE (1.1). In this case, an initial wave spreading is governed by the quantum nonlinear Schrödinger equation

$$i\tilde{h}\partial_t\hat{\psi} = \hat{H}_0\hat{\psi} + \beta|\hat{\psi}|^2\hat{\psi}\,, \tag{2.1}$$

where $[\hat{\psi}(x), \hat{\psi}(x')] = \delta(x - x')$. The Hamiltonian $\hat{H}_0$ corresponds to the Anderson localization problem $\hat{H}_0\Psi_{\omega_k}(x) = \omega_k\Psi_{\omega_k}(x)$, where $\Psi_{\omega_k}(x)$ are the AMs, presented in Introduction. One should recognize that the classical NLSE (1.1) is independent of the Planck constant. Therefore, for the quantum analysis, an effective dimensionless Planck constant $\tilde{h}$ is introduced explicitly. Projecting Eq. (2.1) on the basis of the AMs

$$\hat{\psi}(x,t) = \sum_{\omega_k}\hat{C}_{\omega_k}(t)\Psi_{\omega_k}(x) \equiv \sum_k \hat{C}_k(t)\Psi_k(x)\,, \tag{2.2}$$

we obtain a system of equations for the operators of the expansion $\hat{C}_k$, where $[\hat{C}_k, \hat{C}_l^{\dagger}] = \delta_{k,l}$. This system of the Heisenberg equations $i\tilde{h}\dot{\hat{C}}_k = [\hat{C}_k, , \hat{H}]$ is governed by the quantum Hamiltonian

$$\hat{H} = \sum_k \tilde{h}\omega_k\hat{C}_k^{\dagger}\hat{C}_k + \frac{\tilde{h}^2\beta}{2}\sum_{\mathbf{k}} A_{\mathbf{k}}\hat{C}_{k_1}^{\dagger}\hat{C}_{k_2}^{\dagger}\hat{C}_{k_3}\hat{C}_{k_4}\,, \tag{2.3}$$

where $A(\mathbf{k}) \equiv A_{k_2,k_3}^{k,k_1}$ is an overlapping integral of the four AMs:

$$A_{k_2,k_3}^{k,k_1} = \int \Psi_k(x)\Psi_{k_1}(x)\Psi_{k_2}(x)\Psi_{k_3}(x)dx\,. \tag{2.4}$$

For $\tilde{h} = 0$, Eq. (2.3) reduces to the classical NLSE (1.1). The linear frequency is shifted by the nonlinear term due to the commutation $\omega_k \to \omega_k + \tilde{h}\beta A_0$, where $A_0 = A_{kk}^{kk}$ is the diagonal overlapping integral in Eq. (2.4).

2.1. Phase space dynamics and the Liouville operator

For the quantum mechanical analysis we use a technique of mapping the Heisenberg equation of motion on a basis of the coherent states [30–32]. At the initial moment $t = 0$, one introduces the coherent states vector $|\mathbf{a}\rangle = \prod_q |a_q\rangle$ as the product of eigenfuctions of annihilation operators $\hat{a}_q = \hat{C}_q(t = 0)$, such that $\hat{a}_q|a_q\rangle = a_q|a_q\rangle$ and correspondingly $\hat{a}_q|\mathbf{a}\rangle = a_q|\mathbf{a}\rangle$, where also the coherent state is constructed from a vacuum state $|0\rangle$

$$|a_q\rangle = \exp[a_q\hat{a}_q^{\dagger} - a_q^*\hat{a}_q]|0\rangle\,, \qquad \hat{a}_q|0\rangle = 0\,. \tag{2.5}$$

Introducing c-functions

$$\mathcal{C}_q(t) = \langle\mathbf{a}|\hat{C}_q(t)|\mathbf{a}\rangle = \mathcal{C}_q(t|\mathbf{a}^*, \mathbf{a})\,, \tag{2.6}$$

one maps the Heisenberg equation of motion on the basis $|\mathbf{a}\rangle$ as follows

$$i\tilde{h}\dot{\mathcal{C}}_q(t) = \langle\mathbf{a}|\hat{C}_q(t)\hat{H} - \hat{H}\hat{C}_q(t)|\mathbf{a}\rangle\,. \tag{2.7}$$

Accounting Eqs. (2.5) and (2.6), one obtains the mapping rules

$$\begin{aligned}\langle\mathbf{a}|\hat{C}_q(t)\hat{a}_q^{\dagger}|\mathbf{a}\rangle &= e^{-|a_q|^2}\frac{\partial}{\partial a_q}e^{|a_q|^2}\mathcal{C}_q(t)\,,\\ \langle\mathbf{a}|\hat{a}_q\hat{C}_q(t)|\mathbf{a}\rangle &= e^{-|a_q|^2}\frac{\partial}{\partial a_q^*}e^{|a_q|^2}\mathcal{C}_q(t)\,.\end{aligned} \tag{2.8}$$

The Hamiltonian is the integral of motion $\hat{H}(\{\hat{C}_q^{\dagger}, \hat{C}_q\}) = \hat{H}(\{\hat{a}_q^{\dagger}, \hat{a}_q\})$, therefore the mapping rules (2.8) yield equation of motion (2.7) for $\mathcal{C}_q(t)$ in the closed form

$$i\dot{\mathcal{C}}_q(t) = \hat{K}\mathcal{C}(t)\,, \tag{2.9}$$

where

$$\begin{aligned}\hat{K} &= \frac{1}{\tilde{h}}e^{-\sum_k|a_k|^2}\left[\hat{H}\left(\left\{\frac{\partial}{\partial a_q}, a_q\right\}\right) - \hat{H}\left(\left\{a_q^*, \frac{\partial}{\partial a_q^*}, \right\}\right)\right]e^{\sum_k|a_k|^2}\\ &= \sum_q\left[\omega_q a_q\frac{\partial}{\partial a_q} - c.c\right]\\ &\quad + \frac{\tilde{h}\beta}{2}\sum_{\mathbf{q}}A_{\mathbf{q}}\left[2a_{q_1}a_{q_2}a_{q_3}^*\frac{\partial}{\partial a_{q_4}} + a_{q_1}a_{q_2}\frac{\partial}{\partial a_{q_3}}\frac{\partial}{\partial a_{q_4}} - c.c\right].\end{aligned} \tag{2.10}$$

In the limit $\tilde{h} \to 0$, the second derivative terms vanish, $\tilde{h}a_{q_1}a_{q_2}\frac{\partial}{\partial a_{q_3}}\frac{\partial}{\partial a_{q_4}} \to 0$, and the operator (2.10) reduces to the classical Liouville operator [10].

2.2. Liouville equation

To construct a quantum kinetic equation, or master equation, we consider the density operator $\hat{\rho} = |\hat{\psi}|^2$ and map it on the basis of the coherent states $|\mathbf{a}\rangle$ in Eq. (2.5), such that

$$\mathcal{P}(t) \equiv \mathcal{P}(\mathbf{a}^*, \mathbf{a}, t) = \langle\mathbf{a}|\hat{\rho}(t)|\mathbf{a}\rangle\,. \tag{2.11}$$

Therefore, from the mapping rules one obtains the Liouville equation for the mean probability density

$$\partial_t\mathcal{P}(t) = \hat{L}\mathcal{P}(t)\,. \tag{2.12}$$

Here the quantum "Liouville" operator is determined in Eq. (2.10)

$$\hat{L} = -i\hat{K}\,. \tag{2.13}$$

The initial condition is constructed as a superposition

$$\mathcal{P}(x, t = 0) = \mathcal{P}_0(x) = \sum_{k,k'}F_{k,k'}^{(0)}\Psi_k(x)\Psi_{k'}^*(x)\,, \tag{2.14}$$

where $F_{k,k'}^{(0)} = a_k a_{k'}^*$. Eventually, we obtain that the quantum NLSE (2.1) is replaced by the Liouville Eq. (2.12), which is the *linear* equation with a formal solution in the exponential form

$$\mathcal{P}(x,t) = e^{\hat{L}t}\mathcal{P}_0(x) = \sum_{k,k'}\Psi_k(x)\Psi_{k'}(x) = \sum_{n=0}^{\infty}\left[\frac{t^n}{n!}\hat{L}^n\right]F_{k,k'}^{(0)}\,. \tag{2.15}$$

The Liouville operator can be considered as a sum of two operators $\hat{L} = \hat{L}_0 + \hat{L}_1$, where $\hat{L}_0$ corresponds to the dynamics inside a cluster, determined by the overlapping integrals of the order of 1, while $\hat{L}_1$ corresponds to jumps between these clusters due to the exponentially small overlapping integrals. Thus considering this dynamics in the "interaction picture", where $\hat{L}_1(t) = e^{-\hat{L}_0 t}\hat{L}_1 e^{\hat{L}_0 t}$, one obtains