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Some combinatorial aspects of discrete non-linear population dynamics

Nicolas Grosjean, Thierry Huillet*

Laboratoire de Physique Théorique et Modélisation CNRS-UMR 8089 et Université de Cergy-Pontoise, 2 Avenue Adolphe Chauvin, 95302, Cergy-Pontoise, France

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ABSTRACT

Motivated by issues arising in population dynamics, we consider the problem of iterating a given analytic function a number of times. We use the celebrated technique known as Carleman linearization that turns (for a certain class of functions) this problem into simply taking the power of a real number. We expand this method, showing in particular that it can be used for population models with immigration, and we also apply it to the famous logistic map. We also are able to give a number of results for the invariant density of this map, some being related to the Carleman linearization.

1. Introduction

We consider simple 1-dimensional discrete-time dynamics: $x_{n+1} = \phi(x_n),\ x_0 = x,$ with the evolution mechanism $\phi(\cdot)$ being an analytic function. Our main interest in these problems arises from population dynamics models describing the temporal evolution of some population with size $x_n \geq 0$. We first assume $\phi(0) = 0$ (no immigration). Such non-linear models are amenable to a Carleman linearization giving x_n from the initial condition x in terms of the nth power of some upper-triangular infinite-dimensional transfer matrix which can be diagonalized. Equivalently, ϕ is h-conjugate to the linear map λx for some Carleman function h, would $\lambda = \phi'(0) \neq \{-1, 0, 1\}$. The coefficients of h, as a power series in x, are obtained from the left eigenvector of P with the eigenvalue λ. The Carleman linearization technique goes back the60' [7,9,10,12]. When $\lambda = 1$ (the critical case), we give the linear Carleman representation of x_n, using a 'Jordanization' technique. Special such models arising in population dynamics are defined and investigated.We next consider the problem of computing the invariant density (and its support)of the dynamics in a chaotic population model regime, including quadratic maps. The study of the invariant measures of quadratic and related maps has a very long story starting in the 70' [2–4,11,16]. We show that in some special cases, the h-conjugate representation of ϕ is useful for that purpose. We illustrate our point of view on the celebrated logistic population model $\phi(x) = rx(1-x)$. Next we consider $\phi_0(x) = c + \phi(x)$,modeling some population dynamics with immigration $c > 0$. In the presence of a fixed point for ϕ_0, such models are also Carleman linearizable; equivalently, ϕ_0 is shown to be g-conjugate now to an affine map for some explicit Carleman function g. As an illustration, we finally deal with the logistic population model with immigration. We develop its intimate relation to a family of companion logistic population models without immigration, the former being obtained from the latter through a suitable affine transformation. We exploit this deep connection to determine under which condition the logistic model with immigration is chaotic or not and, using this observation, we compute in some cases its invariant density.

The precise organization of the paper is as follows:

In Section 2, we recall and develop the Carleman transfer matrix linearization technique, including:

- its link with a conjugate representation of the map ϕ in the case $\phi(0) = 0$ and $\phi'(0) \neq \{-1, 0, 1\}$.
- the application of this scheme to the specific critical case $\phi'(0) = 1$, leading to a method akin to the Jordanization of a matrix.
- the consequences of this construction in terms of the invariant measure of the dynamical system.
- the application of this general setting to a class of specific population evolution models.

In Section 3, using the above tools, we focus on the one-parameter logistic population model. Specifically, we characterize the loci and the types of the divergence of its invariant measure

* Corresponding author. Fax: +330134257500.
E-mail addresses: Nicolas.Grosjean@u-cergy.fr (N. Grosjean), Huillet@u-cergy.fr (T. Huillet).

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and we give a way to compute the disconnected components of its support for some parameter r range. For some values of the parameter r, we show how to compute explicitly the invariant density of the system.

In Section 4, expanding the tools introduced in Section 2 to include maps obeying $\phi_0(0) \neq 0$, we study the effect of adding immigration to population dynamics models, by relating it to a an affine conjugate equivalent of the new mechanism with immigration $\phi_0(0) > 0$. Once again, we apply these results to the logistic map. Our main result on this point is summarized in Fig. 1 showing the values of the parameters (r, c) for which this topological conjugation is admissible. As a consequence, the chaoticity of the logistic model with immigration is revealed by the one of the corresponding model without immigration.

2. Carleman matrix in the triangular case $\alpha_0 = 0$

With α_k, $k \geq 1$, real numbers, let $\phi(x) = \sum_{k\geq 1} \alpha_k x^k$, $\alpha_1 \neq 0$, be some smooth power series defined (convergent) in some neighborhood $x_c^- < x < x_c^+$, of the origin where $-\infty \leq x_c^- < 0 < x_c^+ \leq \infty$[1]. We avoid the trivial linear case $\phi(x) = \alpha_1 x$. We shall let $I_c = (x_c^-, x_c^+)$ be the interval of convergence. Note $\alpha_0 = 0$. Consider the dynamical system

$$x_{n+1} = \phi(x_n), x_0 = x. \tag{1}$$

Define the infinite-dimensional (Carleman) upper-triangular matrix

$$P(k, k') = [x^{k'}]\phi(x)^k, k' \geq k \geq 1. \tag{2}$$

By Faà di Bruno formula (see e.g. [5], Tome 1, p. 148), with $\widehat{B}_{k,l}(\alpha_1, \alpha_2, \ldots)$ the (ordinary) Bell polynomials in the coefficients $\alpha_k := [x^k]\phi(x)$ of $\phi(x)$,

$$P(k, k') = \widehat{B}_{k',k}(\alpha_1, \alpha_2, \ldots, \alpha_{k'-k+1}) = k! \sum_{c_l}^{**} \prod_{l=1}^{k'-k+1} \frac{\alpha_l^{c_l}}{c_l!}, \tag{3}$$

where the last double-star summation runs over the integers $c_l \geq 0$ such that $\sum_{l=1}^{k'-k+1} c_l = k$ and $\sum_{l=1}^{k'-k+1} lc_l = k'$ (there are $p_{k,k'}$ terms in this sum, the number of partitions of k' into k summands). In particular $P(k, k) = \alpha_1^k$ and $P(k, k+1) = k\alpha_2\alpha_1^{k-1}$. P is called the Carleman[2] matrix of ϕ. If for example, $\phi(x) = x - x^2$, $P(k, k') = \widehat{B}_{k',k}(z, -1, 0, \ldots)\,|_{z=1}$, the Hermite polynomials evaluated at $z = 1$. We conclude (see [1,10,12,15] and [9]):

Proposition. With $\mathbf{e}_1' = (1, 0, 0, \ldots)$ and $\mathbf{x}' = (x, x^2, \ldots)$,[3]

$$x_n = \mathbf{e}_1' P^n \mathbf{x} \tag{4}$$

where P is an upper-triangular 'transfer' matrix with $P(k, k) = \alpha_1^k =: \lambda^k$, $k \geq 1$ (the eigenvalues of P).

From (1), x_n is also $x_n = \phi_n(x)$ where ϕ_n is the nth iterate of ϕ by composition and so (4) is an alternative linear representation of x_n. Note

$$\sum_{n\geq 0} \lambda^n x_n = \mathbf{e}_1'(I - \lambda P)^{-1}\mathbf{x},$$

involving the resolvent of P.

Remark. We have

$$\frac{1}{1 - u\phi(x)} = 1 + \sum_{k\geq 1} u^k \phi(x)^k$$

$$= 1 + \sum_{k\geq 1} u^k \sum_{k'\geq k} x^{k'} \left[x^{k'}\right]\phi(x)^k$$

$$= 1 + \sum_{k\geq 1} u^k \sum_{k'\geq k} x^{k'} P\left(k, k'\right)$$

$$= 1 + \sum_{k'\geq 1} x^{k'} \sum_{k=1}^{k'} u^k \left[x^{k'}\right]\phi(x)^k$$

$$\sum_{k=1}^{k'} u^k P\left(k, k'\right) = [x^{k'}]\frac{1}{1 - u\phi(x)}.$$

With $\phi_n(x) = x_n$, the nth iterate of ϕ and $\mathbf{u}' := (u, u^2, \ldots)$

$$\frac{1}{1 - u\phi_n(x)} = 1 + \sum_{k\geq 1} u^k \sum_{k'\geq k} x^{k'} P^n(k, k')$$

$$= 1 + \sum_{k'\geq 1} x^{k'} \sum_{k=1}^{k'} u^k P^n(k, k')$$

$$= 1 + \mathbf{u}' P^n \mathbf{x}.$$

Taking the derivative with respect to u at $u = 0$ gives $\phi_n(x) = x_n = \mathbf{e}_1' P^n \mathbf{x}$. Taking the kth derivative with respect to u at $u = 0$ gives $x_n^k = \mathbf{e}_k' P^n \mathbf{x}$. We conclude:

Proposition. If $\psi(x) = \sum_{k\geq 1} \psi_k x^k$ is some smooth observable, defining $\psi' = (\psi_1, \psi_2, \ldots)$, therefore

$$\psi(x_n) = \sum_k \psi_k \mathbf{e}_k' P^n \mathbf{x} = \psi' P^n \mathbf{x}. \tag{5}$$

This generalizes (4).

By Cauchy formula, whenever ϕ is defined on the unit circle, we also have the Fourier representation

$$P\left(k, k'\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{ik'\theta} \phi(e^{-i\theta})^k d\theta.$$

Chaos for (1) is sometimes characterized by the positivity of its Lyapounov exponent defined by

$$\lambda(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log|\phi\prime(x_n)|$$

for almost all x. Considering the sensitivity to the initial condition problem, we have $J_{n+1} := dx_{n+1}/dx = \phi'(x_n)dx_n/dx = \phi'(x_n)J_n$. Therefore $|J_N| = (\prod_{n=0}^{N-1} |\phi'(x_n)|)$ and $\lambda_N(x) := \frac{1}{N}\log|J_N| \to \lambda(x)$. Letting $\mathbf{x}' = (x, x^2, \ldots)$, $\overline{\mathbf{x}}' := (1, x, x^2, \ldots)$ and $D :=$ diag $(1, 2, 3, \ldots)$ so that $d\mathbf{x}/dx = D\overline{\mathbf{x}}$, we observe

$$J_N := dx_N/dx = \mathbf{e}_1' P^N D\overline{\mathbf{x}}.$$

2.1. The case $|\lambda| \neq 1$

Suppose $\lambda := \alpha_1 \neq \pm 1$ and let $\mathbf{v}_k' P = \lambda^k \mathbf{v}_k'$, define the left-row-eigenvector $\mathbf{v}_k'$ of P associated to the eigenvalue $\lambda_k := \lambda^k$. Then, with $k' > k \geq 1$,

$$\mathbf{v}_k(k') = (\lambda^k - \lambda^{k'})^{-1} \sum_{l=1}^{k'-1} P(l, k')\mathbf{v}_k(l)$$

gives the entries $\mathbf{v}_k(k')$, $k' > k \geq 1$, of $\mathbf{v}_k'$ by recurrence; and $\mathbf{v}_k(k)$ can be left undeterminate. Developing, for $k' > k$, we get

Proposition. Suppose $\lambda = \alpha_1 \neq 1$, then

$$\mathbf{v}_k\left(k'\right) = \mathbf{v}_k(k) \sum_{j=2}^{k'-k+1} \sum_{d_1+\cdots+d_{j-1}=k'-k}^{*} \prod_{l=1}^{j-1} \times \frac{P(d_1 + \cdots + d_{l-1} + k, d_1 + \cdots + d_l + k)}{\lambda^k - \lambda^{d_1+\cdots+d_l+k}}, \text{ or}$$

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[1] We also assume that ϕ is absolutely convergent with radius of convergence $0 < r_c \leq \min(-x_c^-, x_c^+)$

[2] Carleman matrices are easily seen to be the transpose of Bell matrices.

[3] Throughout, a boldface variable, say $\mathbf{x}$, will represent a column-vector and its transpose, say $\mathbf{x}'$, will be a row-vector.