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# Strange attractors in periodically kicked predator - prey system with discrete and distributed delay☆

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ABSTRACT

In this paper, the behavior of rank one strange attractors in a predator - prey system with discrete and distributed delay is studied. This investigation is conducted from the perspective of a recent chaos theory of rank one maps developed by us for delayed systems. We specifically choose the well - known predator - prey system with discrete and distributed delay to demonstrate the applicability of the theory to the case of ecological systems, and prove the appearance of chaotic behavior under reasonable conditions. The results of the numerical simulations presented are in close agreement with the expectations of the theory.



## 1. Introduction

The development of the theory of rank one attractors has a long and celebrated history. It originated from Jakobson's theory on quadratic maps [1] and Benedicks - Carleson's theory on Hénon maps [2,3]. This analytic machinery was recently generalized and further developed into a theory of rank one attractors by Wang and Young [4]. The term rank one chaos, as it is used in this paper, indicates a number of precisely defined dynamical properties that together imply sustained, observable chaos [5]. These properties include:($a$) positive Lyapunov exponents starting from almost all initial conditions in the basin; and ($b$) cohesive statistical properties represented by the existence of SRB measures [6–8]. In [9,10], this theory is applied to the study of periodically kicked systems of ordinary differential equations. In [11], we developed rank one theory for delayed differential equation. Positive Lyapunov exponents play a important role in the theory of rank one chaos. In 2007, G. Leonov et al. [12] discussed largest Lyapunov exponent between the time-varying original system and its linear system of the first approximation, they point out that the systems with constant or periodic coefficients are regular. In this paper, we introduce a predator - prey system with discrete and distributed delay by adding an external periodic force and show the existence of rank one attractors.

In [13], Song and Peng formulate a logistic model with discrete and distributed delay as follows:

$$\dot{x}(t) = rx(t)[1 - a_1x(t-\tau) - a_2\int_{-\infty}^{t} f(t-s)x(s)ds], \tag{1}$$

where $r, a_1, a_2, \tau > 0$. The function $f$ in Eq. (1) is called the delayed kernel, which is the weight given to the population $t$ time units ago, and it satisfies $f(t) \geq 0$ for all $t \geq 0$ together with the normalization condition $\int_0^\infty f(s)ds = 1$. Taking the delay $\tau$ as a parameter, they investigate the effect of the delay $\tau$ on the dynamics of Eq. (1). More specifically, it shows that the stability switches and a Hopf bifurcation occur when the delay $\tau$ passes through a critical value. In [14], Dodd considered the periodic orbits arising from Hopf bifurcations of the following delayed system:

$$\begin{cases} \dot{x}(t) = r_1x(t)[1 - a_{11}x(t) - a_{12}\int_0^\infty y(t-u)dh_1(u)], \\ \dot{y}(t) = r_2y(t)[-1 + a_{21}\int_0^\infty x(t-u)dh_2(u)]. \end{cases} \tag{2}$$

They showed that under certain conditions stable periodic solutions arising from Hopf bifurcations at different critical values of the parameters can exist together. In [15], Liu et al. researched the following impulsive differential equation with a fixed moment:

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t) - a_{12}y(t)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t)], \quad t \neq nT, \\ \Delta x(t) = -p_1x(t), \\ \Delta y(t) = -p_2y(t) + \mu, \quad t = nT, \end{cases} \tag{3}$$

where $\Delta x(t) = x(t^+) - x(t), \Delta y(t) = y(t^+) - y(t) (0 \leq p_1 < 1, 0 \leq p_2 < 1)$ represents the fraction of pest (predator) which dies due to the pesticide, $\mu \geq 0$ is the release amount of predator at $t = nT, n \in Z_+$ and $T$ is the period of the impulsive effect.

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In this paper, we consider a predator-prey system with a discrete and distributed delay as follows:

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t) - a_{12}y(t-\tau)] + \varepsilon P_T x(t), \\ \dot{y}(t) = y(t)[-r_2 + a_{21}\int_{-\infty}^{t} F(t-s)x(s)ds], \end{cases} \tag{4}$$

where $r_1, r_2, a_{11}, a_{12}, a_{21}$ and $\tau$ are real positive parameters, $P_T = \Sigma_{n=-\infty}^{\infty}\delta(t-nT)$. Following the ideal of Cushing [16], we take the delay kernel as $F(s) = \gamma e^{-\gamma s}$, $\gamma > 0$. The stability of equilibrium, the bifurcating periodic solutions and the rank one chaos of the periodically kicked delayed system are investigated.

In Section 2, we present the preliminaries about the rank one theory of delayed differential equation. In Section 3, we consider rank one chaos for the periodically kicked predator - prey system with discrete and distributed delay. In Section 4, numerical simulations are presented. Conclusions are given in Section 5.

## 2. Preliminaries

To properly motivate the studies presented in this paper, we first give a brief overview on the studies of rank one strange attractors in delayed differential equation [11].

Considering a nonlinear delayed differential equation:

$$\frac{d\mathbf{u}}{dt} = L_\mu \mathbf{u}_t + f_\mu(\mathbf{u}_t) + \varepsilon P_T \Phi(\mathbf{u_t}), \tag{5}$$

where $\mathbf{u}_t(\theta) = \mathbf{u}(t+\theta), \theta \in [-r, 0]$ for $r > 0$, $L_\mu : C[-r, 0] \to \mathbb{R}^n$ is a linear operator, $f_\mu : \mathrm{C}[-r, 0] \to \mathbb{R}^n$ is a nonlinear term satisfying $f_\mu(\mathbf{0}) = \mathbf{0}, D_\mathbf{u} f_\mu(\mathbf{0}) = \mathbf{0}$. $f_\mu$ and $L_\mu$ depend on $\mu \in \mathbb{R}$ analytically for $|\mu|$ is sufficiently small, and $P_T = \Sigma_{n=-\infty}^{\infty}\delta(t-nT)$. When $\varepsilon = 0$, the undisturbed system is

$$\frac{d\mathbf{u}}{dt} = L_\mu \mathbf{u}_t + f_\mu(\mathbf{u}_t). \tag{6}$$

For linear system $\dot{\mathbf{u}} = L_\mu \mathbf{u}_t$, there is an $n \times n$ matrix $\eta(\cdot, \mu) : [-r, 0] \to \mathbb{R}^{n^2}$, whose elements are of bounded variation such that for any $\phi \in \mathrm{C}[-r, 0]$

$$L_\mu \phi = \int_{-r}^{0} d\eta(\theta, \mu)\phi(\theta). \tag{7}$$

Let the spectral set of $L_\mu$

$$\sigma(\mu) = \{\lambda \mid det(\lambda I - L_\mu e^{\lambda\theta} I) = 0\}$$

satisfies

$(B1)$ There is a simple conjugated pair

$$\lambda_{1,2} = a(\mu) \pm i\omega(\mu)$$

such that $a(0) = 0, \omega(0) = \omega_0 > 0$, $(d/d\mu)a(0) \neq 0$, and there exists $c > 0$ such that for any $\lambda \in \sigma(\mu)$, $\lambda \neq \lambda_{1,2}$, $Re(\lambda_i) < -c, i \geq 3$.

Then the flow on the central manifold $W^c$ of system (6) can be written by using Hassard's method [17] in the following normal form

$$\dot{z} = iw_0 z + g(z, \bar{z}) = iw_0 z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + \frac{g_{21}}{2}z^2\bar{z} + \ldots, \tag{8}$$

at $\mu = 0$. Let

$$k_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}. \tag{9}$$

Suppose that

$(B2)$ $Re(k_1(0)) < 0$.

Then we know that system (6) has a supercritical Hopf bifurcation near the equilibrium.

We define for $\phi \in \mathrm{C}[-r, 0]$

$$A(\mu)\phi = \begin{cases} \frac{d\phi}{d\theta}, & \theta \in [-r, 0), \\ \int_{-r}^{0} d\eta(s, \mu)\phi(s) = L_\mu\phi, & \theta = 0, \end{cases} \tag{10}$$

and

$$R\phi = \begin{cases} 0, & \theta \in [-r, 0), \\ f_\mu(\phi) + \varepsilon P_T\Phi(\phi), & \theta = 0. \end{cases} \tag{11}$$

Since $\frac{d\mathbf{u}_t}{d\theta} = \frac{d\mathbf{u}_t}{dt}$, Eq. (5) becomes

$$\dot{\mathbf{u}}_t = A(\mu)\mathbf{u}_t + R\mathbf{u}_t. \tag{12}$$

For $\theta = 0$ Eq. (12) is Eq. (5).

Following Hassard's method. Let

$$z(t) = \langle q^*(\theta), u_t(\theta)\rangle,$$

where $q^*(\theta)$ is a eigenvector corresponding to eigenvalue $-i\omega_0$ of $A^*$ which is the adjoint operator of $A(0)$.

When $\mu = 0$, let $A = A(0)$ and define

$$\begin{aligned} W(t, \theta) &= u_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta) \\ &= u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\} \\ &= W(z(t), \bar{z}(t), \theta). \end{aligned} \tag{13}$$

At $\theta = 0$,

$$\begin{aligned} \dot{z} &= i\omega_0 z + \bar{q}^*(0)f_0(W(z, \bar{z}, 0) + 2Rezq(0)) \\ &\quad + \varepsilon P_T\bar{q}^*(0)\Phi(W(z, \bar{z}, 0) + 2Rezq(0)) \\ &= i\omega_0 z + g(z, \bar{z}) + \varepsilon P_T\bar{q}^*(0)\Phi(W(z, \bar{z}, 0) + 2Rezq(0)). \end{aligned} \tag{14}$$

We can calculate

$$\begin{aligned} \dot{W} &= \dot{u_t} - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= AW + H(z, \bar{z}, 0). \end{aligned}$$

Then Eq. (12) can be written as

$$\begin{cases} \dot{z} = i\omega_0 z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + g_{21}z^2\bar{z} + \cdots \\ \qquad + \varepsilon P_T\bar{q}^*(0)\Phi(W(z, \bar{z}, 0) + 2Rezq(0)), \\ \dot{W} = AW + H(z, \bar{z}, 0). \end{cases} \tag{15}$$

Let $W \in \mathcal{B}$, where $\mathcal{B}$ is a Banach space. Let $z = x + iy$ in (15). Define

$$\begin{aligned} \Psi_x(x, y) &= Re\{\bar{q}^*(0)\Phi(W(x, y, 0) + 2Re(x + iy)q(0))\}, \\ \Psi_y(x, y) &= Im\{\bar{q}^*(0)\Phi(W(x, y, 0) + 2Re(x + iy)q(0))\}. \end{aligned} \tag{16}$$

We further let $x = \cos\theta$, $y = \sin\theta$, $W = \mathbf{0}$ in (16), then $\{\hat{\mathbf{s}}_0 = (\cos\theta, \sin\theta, \mathbf{0}) \in S \times \mathcal{B},\ \theta \in [0, 2\pi)\}$ is a unit circle in $(x, y)$ - plane in $(x, y, W)-$space. Define

$$\varphi(\theta) = \cos\theta\Psi_x(\hat{s}_0) + \sin\theta\Psi_y(\hat{s}_0). \tag{17}$$

The time$-T$ map of Eq. (5) is denoted by $F_{\mu,\,\varepsilon,\,T}$, where $\mu$ is the bifurcation parameter of the unperturbed Eq. (6) and $\varepsilon$, $T$ are the parameters of forcing. Assume that

(a) $(B1) - (B2)$ hold for Eq. (5);
(b) $\varphi(\theta)$ in Eq. (17) is a Morse function.

Then we obtain [11]

**Lemma 1.** *Assume that (a) - (b) hold. Regard the period T of the forcing as a parameter and denote $F_T = F_{\mu,\varepsilon,T}$. Then there exists a constant $K_2$, determined exclusively by $\varphi(\theta)$, such that if*

$$\left|\varepsilon\frac{Imk_1(0)}{Rek_1(0)}\right| > K_2,$$

*then there exists a positive measure set $\Delta \subset (\mu^{-1}, \infty)$ for T, so that for $T \in \Delta$, $F_T$ has a strange attractor $\Lambda$ admitting no periodic sinks. This is to say that there exists an open neighborhood U of $\Lambda$ such that $F_T$ has a positive Lyapunov exponent for Lesbegue almost every point in U. Furthermore, $F_T$ admits an ergodic SRB measure, with respect to which almost every point of U is generic.*