Chaos, Solitons and Fractals 93 (2016) 187–193

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Randomly orthogonal factorizations in networks[☆]

Sizhong Zhou[a,*], Lan Xu[b], Yang Xu[c]

[a] School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, PR China
[b] Department of Mathematics, Changji University, Changji, Xinjiang 831100, PR China
[c] Department of Mathematics, Qingdao Agricultural University, Qingdao, Shandong 266109, PR China

ARTICLE INFO

Article history:
Received 10 August 2016
Revised 28 October 2016
Accepted 31 October 2016

MSC:
05C70

Keywords:
Networks
Graph
Subgraph
(g, f)-factor
r-orthogonal factorization

ABSTRACT

Let m, r, k be three positive integers. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let f: $V(G) \to N$ be a function such that $f(x) \geq (k+2)r-1$ for any $x \in V(G)$. Let H_1, H_2, …, H_k be k vertex disjoint mr-subgraphs of a graph G. In this paper, we prove that every $(0, mf-(m-1)r)$-graph admits a $(0, f)$-factorization randomly r-orthogonal to each H_i $(i = 1, 2, \ldots, k)$.

1. Introduction

Many real-world networks can conveniently be modelled by graphs or networks. Examples include the World Wide Web with nodes and links modelling Web pages and hyperlinks between Web pages, respectively, or a communication network with nodes presenting cities, and links corresponding to communication channels, or a railroad network with nodes and links modelling railroad stations and railways between two stations, respectively. Orthogonal factorizations in graphs or networks have attracted a great deal of attention [3,5–9,11–13] due to their applications in combinatorial design, network design, circuit layout, and so on. For example, Euler [2] first found that a pair of orthogonal Latin squares of order n is equivalent to two orthogonal 1-factorizations of a complete bipartite graph $K_{n,n}$. A Room square of order $2n$ is related to the orthogonal 1-factorization of a complete graph K_{2n}, which was first presented by Horton [4]. Many other applications in this field can be found in a current survey [1]. It is well known that a graph can represent a network. Vertices of the graph correspond to nodes of the network, and edges of the graph correspond to links between the nodes in the network. Henceforth we use the term *graph* instead of *network*.

In this paper, all graphs considered will be finite undirected graphs without loops or multiple edges. Let G be a graph. We denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For arbitrary $x \in V(G)$, $d_G(x)$ denotes the degree of x in G. Let g, f: $V(G) \to N$ be two functions such that $g(x) \leq f(x)$ for arbitrary $x \in V(G)$. A spanning subgraph F of a graph G with $g(x) \leq d_F(x) \leq f(x)$ for every $x \in V(G)$ is called a (g, f)-factor of G. Especially, G is said to be a (g, f)-graph if G itself is a (g, f)-factor. A (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ of a graph G is a partition of $E(G)$ into edge-disjoint (g, f)-factors F_1, F_2, …, F_m. A subgraph H of a graph G is called an m-subgraph if H admits m edges in total. Let H be an mr-subgraph of a graph G and $F = \{F_1, F_2, \ldots, F_m\}$ be a (g, f)-factorization of a graph G. A (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ is r-orthogonal to H if $|E(F_i) \cap E(H)| = r$ for $1 \leq i \leq m$. We say that a graph G admits (g, f)-factorizations randomly r-orthogonal to H if for arbitrary partition $\{A_1, A_2, \ldots, A_m\}$ of $E(H)$ satisfying $|A_i| = r$, there is a (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ of a graph G with $A_i \subseteq E(F_i)$, $i = 1, 2, \ldots, m$. It is easy to see that randomly 1-orthogonal is equivalent to 1-orthogonal and 1-orthogonal is also said to be orthogonal.

Alspach et al. [1] introduced orthogonal factorizations of graphs and presented the following question: Given a subgraph H of a graph G, does there exist a factorization F of a graph G of certain type orthogonal to H?

☆ This work is supported by the National Natural Science Foundation of China (Grant No. 11371009, 11501256, 61503160) and the National Social Science Foundation of China (Grant No. 14AGL001) and the Natural Science Foundation of Xinjiang Province of China (Grant No. 2015211A003), and sponsored by 333 Project of Jiangsu Province.

* Corresponding author.
E-mail address: zsz_cumt@163.com (S. Zhou).

http://dx.doi.org/10.1016/j.chaos.2016.10.019
0960-0779/

Li and Liu [6] justified that every $(mg+m-1, mf-m+1)$-graph G has a (g, f)-factorization orthogonal to arbitrary given m-subgraph. Li, Chen and Yu [5] proved that for every $(mg+k, mf-k)$-graph G and its arbitrary given k-subgraph H, there exists a subgraph R which has a (g, f)-factorization orthogonal to H. Li and Liu [7] verified that for every $(mg+kr, mf-kr)$-graph G and its arbitrary given kr-subgraph H, G includes a subgraph R such that R admits a (g, f)-factorization r-orthogonal to H. Liu and Zhu [9] showed that every bipartite $(mg+m-1, mf-m+1)$-graph has (g, f)-factorizations randomly r-orthogonal to any given mr-subgraph if $f(x) \geq g(x) \geq r$ for arbitrary $x \in V(G)$. Liu and Long [8] showed that every $(mg+m-1, mf-m+1)$-graph G admits a (g, f)-factorization randomly r-orthogonal to any given subgraph H with mr edges if $f(x) \geq g(x) \geq 2r-1$ for arbitrary $x \in V(G)$. Zhou and Zong [13] proved that every $(0, mf-(m-1)r)$-graph G has a $(0, f)$-factorization randomly r-orthogonal to any given mr-subgraph H if $f(x) \geq 3r-1$ for each $x \in V(G)$.

Theorem 1 (Zhou and Zong [13]). *Let G be a $(0, mf-(m-1)r)$-graph, and let f be an integer-valued function defined on $V(G)$ such that $f(x) \geq 3r-1$ for all $x \in V(G)$, and let H be an mr-subgraph of G. Then G has a $(0, f)$-factorization randomly r-orthogonal to H.*

In this paper, we consider the more general problem: Given k vertex-disjoint mr-subgraphs $H_1, H_2, \ldots, H_k$ of a graph G, does there exist a factorization F randomly r-orthogonal to every H_i $(1 \leq i \leq k)$?

We show that this question above is true by the following theorem which is an extension of Theorem 1.

Theorem 2. *Let m, r, k be three positive integers, and let G be a $(0, mf-(m-1)r)$-graph, and let f be an integer-valued functions defined on $V(G)$ satisfying $f(x) \geq (k+2)r-1$ for arbitrary $x \in V(G)$. Let $H_1, H_2, \ldots, H_k$ be k vertex disjoint mr-subgraphs of G. Then G has a $(0, f)$-factorization randomly r-orthogonal to every H_i, $1 \leq i \leq k$.*

2. Preliminary lemmas

Let S and T be two disjoint vertex subsets of G. We denote the set of edges with one end in S and the other in T by $E_G(S, T)$, and write $e_G(S,T) = |E_G(S,T)|$. For $S \subset V(G)$ and $A \subset E(G)$, $G[S]$ and $G[A]$ are two subgraphs of a graph G induced by S and A, respectively. We write $G-S = G[V(G) \setminus S]$ and $G-A = G[E(G) \setminus A]$. For any function φ defined on $V(G)$, we write $\varphi(X) = \sum_{x \in X} \varphi(x)$ and $\varphi(\emptyset) = 0$, where $X \subseteq V(G)$. Especially, $d_G(X) = \sum_{x \in X} d_G(x)$.

Let g, f: $V(G) \to N$ be two functions defined on $V(G)$ with $g(x) \leq f(x)$ for every $x \in V(G)$. If C is a component of $G-(S \cup T)$ with $g(x) = f(x)$ for every $x \in V(C)$, then we call C is odd or even in terms of $e_G(T, V(C)) + f(V(C))$ being odd or even, respectively. We write $h_G(S, T)$ for the number of the odd components of $G-(S \cup T)$. In 1970 Lovász [10] used the symbol $\delta_G(S, T; g, f)$ for the expression $f(S) + d_{G-S}(T) - h_G(S,T) - g(T)$, and found that $\delta_G(S, T; g, f) \geq 0$ is a necessary and sufficient condition for the existence of a (g, f)-factor in a graph G.

Lemma 2.1 (Lovász [10]). *Let G be a graph, and g, f: $V(G) \to N$ be two functions defined on $V(G)$ with $g(x) \leq f(x)$ for any $x \in V(G)$. Then G admits a (g, f)-factor if and only if*

$$\delta_G(S, T; g, f) \geq 0$$

for arbitrary two disjoint vertex subsets S and T of G.

Note that if $g(x) < f(x)$ for all $x \in V(G)$ then $h_G(S,T) = 0$. Let S and T be two disjoint vertex subsets of G, and let E_1 and E_2 be two disjoint edge subsets of G. Define

$$D = V(G) - (S \cup T), \quad E(S) = \{xy \in E(G) : x, y \in S\}$$

and

$$E(T) = \{xy \in E(G) : x, y \in T\},$$

Set

$$E_1' = E_1 \cap E(S), \quad E_1'' = E_1 \cap E_G(S, D),$$

$$E_2' = E_2 \cap E(T), \quad E_2'' = E_2 \cap E_G(T, D),$$

$$\alpha(S, T; E_1, E_2) = 2|E_1'| + |E_1''|, \quad \beta(S, T; E_1, E_2) = 2|E_2'| + |E_2''|.$$

Under without ambiguity, $\alpha(S, T; E_1, E_2)$ and $\beta(S, T; E_1, E_2)$ are written as α and β, respectively.

By using Lemma 2.1, Li and Liu [6] justified the following result, which is very useful for verifying Theorem 2.

Lemma 2.2 (Li and Liu [6]). *Let G be a graph, and g, f: $V(G) \to N$ be two functions defined on $V(G)$ with $0 \leq g(x) < f(x) \leq d_G(x)$ for every $x \in V(G)$. Let E_1 and E_2 be two disjoint edge subsets of G. Then G admits a (g, f)-factor F with $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if*

$$\begin{aligned}\delta_G(S, T; g, f) &= f(S) + d_{G-S}(T) - g(T) \\ &\geq \alpha(S, T; E_1, E_2) + \beta(S, T; E_1, E_2)\end{aligned}$$

for arbitrary two disjoint vertex subsets S and T of G.

In the following, we always assume that G is a $(0, mf-(m-1)r)$-graph, where m, r are two positive integers. For each $(0, f)$-factor F_i and each isolated vertex x of G, we have $d_{F_i}(x) = 0$, which implies that the required condition is satisfied. Therefore, we may assume that G has no isolated vertices. Set

$$p(x) = \max\{0, d_G(x) - (m-1)f(x) + (m-2)r\}$$

and

$$q(x) = \min\{f(x), d_G(x)\}$$

for every $x \in V(G)$. In terms of the definition of $p(x)$ and $q(x)$, it is obvious that $0 \leq p(x) < q(x) \leq f(x)$ for every $x \in V(G)$. For any $x \in V(G)$, we write

$$\triangle_1(x) = \frac{1}{m} d_G(x) - p(x)$$

and

$$\triangle_2(x) = q(x) - \frac{1}{m} d_G(x).$$

Lemma 2.3 (Li and Liu [6]). *For arbitrary two disjoint subsets S and T of $V(G)$, the following equality holds*

$$\delta_G(S, T; p, q) = \Delta_1(T) + \Delta_2(S) + \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S).$$

3. Proof of Theorem 2

Let G be a graph, and let f: $V(G) \to N$ be a function defined on $V(G)$ with $f(x) \geq (k+2)r-1$ for arbitrary $x \in V(G)$. Let $H_1, H_2, \ldots, H_k$ be k vertex-disjoint mr-subgraphs of a graph G. We choose any $A_i \subseteq E(H_i)$ satisfying $|A_i| = r$, $i = 1, 2, \ldots, k$. Set $E_1 = \bigcup_{i=1}^{k} A_i$ and $E_2 = (\bigcup_{i=1}^{k} E(H_i)) \setminus E_1$. It is obvious that $|E_1| = kr$ and $|E_2| = (m-1)kr$. For two disjoint subsets $S, T \subseteq V(G)$, we define $E_1', E_1'', E_2', E_2'', \alpha$ and β as in Section 2. In terms of the definitions of α and β, we obtain

$$\alpha \leq \min\{2kr, r|S|\}$$

and

$$\beta \leq \min\{2(m-1)kr, (m-1)r|T|\}.$$

The definitions of $p(x)$, $q(x)$, $\Delta_1(x)$ and $\Delta_2(x)$ are identical to that in Section 2. In order to justify Theorem 2, we first verify the following lemma which is very useful for proving Theorem 2.