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On groupoid gradings

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ABSTRACT

We introduce the notion of groupoid grading, give some nontrivial examples and prove that groupoid gradings on simple commutative or anti-commutative algebras are necessarily group gradings. We also take advantage of the structure of groupoids to prove some results about groupoid gradings and certain coarsenings of these which turn out to be group gradings. We also study set gradings on arbitrary algebras, by characterizing their homogeneous semisimplicity and their homogeneous simplicity in terms of a property satisfied by the supports of the gradings, and also relate set gradings with groupoid gradings via coarsenings. Finally we study a class of set gradings on $M_n(\mathbb{C})$, the orthogonal gradings, and show that all of them which are fine are necessarily groupoid gradings.

1. Introduction and preliminary definitions

In the literature, group gradings have been intensively studied in the last years, motivated in part by their application in physics, geometry and topology where they appear as the natural framework for an algebraic model [1–11]. In particular, in the field of mathematical physics, they play an important role in the theory of strings, color supergravity, Walsh functions or electroweak interactions [12–17]. Certain advantages of endowing with a grading to an algebra can also be found in [18–20]. However, gradings by means of weaker structures than a group have been considered in the literature just in a slightly way. In the preset paper we wish to study algebras which are graded by means of not necessarily a group but just a groupoid. Finally, we note that a complete review of the state of the art, respect to the theory of graded algebras, can be found in the recent monograph [21].

For any category $\mathcal{C}$, we will denote the class of objects of $\mathcal{C}$ by $\mathrm{Obj}(\mathcal{C})$ and the class of morphisms of $\mathcal{C}$ by $\mathrm{Mor}(\mathcal{C})$. A *groupoid* G is a small category in which every arrow has an inverse. An alternative definition, which can be found for instance in [22], is given in terms of a partially defined multiplication, on a non-empty set G, with some associativity condition and also a suitable invertibility property. When we work with groupoids, the multiplication is understood to be arrow composition (only possible when $f \in \hom(A, B)$ and $g \in \hom(B, C)$. Usually we will identify the arrows of the category with the elements of the groupoid. If as before, $f \in \hom(A, B)$ and $g \in \hom(B, C)$, the product fg in the groupoid is by definition the composition $g \circ f$, that is, $fg := g \circ f$.

We note that, throughout the paper, the product of all of the considered algebras will be denoted by juxtaposition. If we take a groupoid G, then a *groupoid grading* on an algebra A by G is a decomposition of A as a direct sum of linear

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subspaces

$$\Lambda : A = \bigoplus_{f \in G} A_f$$

such that, if $A_f A_g \neq 0$ then fg is defined, that is $f \in \hom(A, B)$, $g \in \hom(B, C)$ for suitable objects, and $A_f A_g \subset A_{fg}$. The *support* of the grading is the set $\Sigma := \{f \in G : A_f \neq 0\}$. Of course, since any group is a groupoid, any group grading is a groupoid grading.

If we have two groupoid gradings $\Lambda_1 : A = \oplus_{g\in G} A_g$ and $\Lambda_2 : A = \oplus_{h\in H} B_h$ on a K-algebra A, we will say that Λ_1 is a *refinement* of Λ_2 when each homogeneous component A_g is contained in one homogeneous component B_h. We will also say that Λ_2 is a *coarsening* of Λ_1. The grading Λ will be called *fine* if it does not admits any proper refinement. Finally, it is also said that Λ_1 and Λ_2 are *equivalent* if there exist a bijection between their supports $\sigma : \Sigma_1 \to \Sigma_2$, and an isomorphism of algebras $\phi : A \to A$ such that $\phi(A_g) = B_{\sigma(g)}$ for any $h \in \Sigma_1$.

In the present work we will also have the opportunity, in Sections 4 and 5, of dealing with the most general concept of grading on an algebra, which is the one of set grading. If A is an arbitrary algebra and I an arbitrary (non-empty) set. It is said that A has a *set grading*, by means of I, if

$$A = \bigoplus_{i\in I} A_i \tag{1}$$

where any A_i is a linear subspace satisfying that for any $j \in I$ either $A_i A_j = 0$ or $0 \neq A_i A_j \subset A_k$ for some (unique) $k \in I$. As usual, the *support* of the grading is the set $\Sigma := \{i \in I : A_i \neq 0\}$, and the concepts of *refinement*, *coarsening*, *fine grading* and *equivalence between gradings* are defined for set gradings in a similar way than for groupoid gradings.

The paper is organized as follows. Section 2 is devoted to present several examples of groupoid gradings, on different algebras, which are not group gradings. In Section 3 we study groupoid gradings on commutative and anti-commutative algebras, (in particular on Jordan and Lie algebras). We recall that maybe the most famous conjecture in the framework of graded algebras says that any set grading on a simple, complex finite-dimensional, Lie algebra is equivalent to an Abelian group grading on the same Lie algebra, [21]. We give in Section 3 a partial proof of this conjecture, by showing that any groupoid grading on a simple commutative or anti-commutative algebra is necessarily a group grading. Section 4 is devoted to study groupoid, (and set), gradings on arbitrary algebras A. That is, A is just a linear space endowed with a bilinear map called the product of A, but any identity (associative, alternative, Lie, Jordan, etc.) for the product is not supposed. We focus on the inner structure of these gradings by proving they admit coarsenings which are also groupoid gradings but with all of their homogeneous components ideals of A. Moreover, we also show that this result also holds when we part from an arbitrary set graded algebra. If the groupoid is furthermore connected, (for instance when A is simple), then the above coarsenings can be taken as (nontrivial) group gradings. Finally, a characterization of the homogeneous semisimplicity and the homogeneous simplicity of a set grading on an arbitrary algebra, in terms of certain property of the support of the grading, is given. The last section, Section 5, is devoted to study a special type of set gradings on (complex) matrix algebras, the orthogonal gradings. These are set gradings which are compatible with the standard involution and inner product of the matrix algebra. We prove that any orthogonal grading on a (complex) matrix algebra which is fine is necessarily a groupoid grading.

2. Groupoid gradings which are not group gradings

In order to give some examples of groupoid gradings which are not group gradings we must recall that a nice way to present a category is by means of a (directed) graph. Recall that such a graph is a 4-tuple $E = (E^0, E^1, s, r)$ where E^0 is a set (the so called set of vertices), E^1 is also a set (the set of edges) and $r, s : E^1 \to E^0$ where for any $f \in E^1$ we say that $s(f)$ is the source of f and $r(f)$ the range of f.

An interesting notion on a graph E is that of a (nontrivial) path: a finite sequence $\lambda := f_1 \cdots f_n$ of edges $f_i \in E^1$ such that $r(f_i) = s(f_{i+1})$. The vertices are usually considered as trivial paths. Also we say that the source of the path λ is the source of f_1 while the range of λ is the range of f_n. The source and range of a trivial path ($u \in E^0$) is the vertex u itself. Furthermore the multiplication of paths λ, μ is also possible if $r(\lambda) = s(\mu)$, in which case it has sense the path $\lambda\mu$. If $u \in E^0$ is a trivial path and λ a path with $s(\lambda) = u$, the multiplication of these path it is defined by $u\lambda = \lambda$. Similarly if $r(\lambda) = v$ then $\lambda v = \lambda$. This multiplication is associative when the compositions have sense.

When one has a graph E it is possible to define a (small) category $\mathcal{C}$ whose objects are the vertices of E, that is, $\mathrm{Obj}(\mathcal{C}) = E^0$ and the arrows of $\mathcal{C}$ are the paths of the category. For given objects $A, B \in \mathrm{Obj}(\mathcal{C})$, we define $\hom_{\mathcal{C}}(A, B)$ to be the set of all paths whose source is A and whose range is B. In particular there is an arrow $1_A \in \hom(A, A)$, in fact, we define 1_A as the trivial path, the vertex A. The multiplication

$$\hom(A, B) \times \hom(B, C) \to \hom(A, C)$$

that we need in order to have a category, is given by juxtaposition of paths.

So for instance, consider the graph: