Journal of Geometry and Physics 123 (2018) 98–126

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Projective limits of state spaces III. Toy-models[☆]

Suzanne Lanéry[a,b,*], Thomas Thiemann[a]

[a] Institute for Quantum Gravity, Friedrich-Alexander Universität Erlangen-Nürnberg, Staudtstraße 7/B2, 91058 Erlangen, Germany
[b] Laboratoire de Mathématiques et Physique Théorique, Université François-Rabelais de Tours, UFR Sciences et Techniques, Parc de Grandmont, 37200 Tours, France

ARTICLE INFO

Article history:
Received 6 May 2016
Received in revised form 10 August 2017
Accepted 12 August 2017
Available online 8 September 2017

MSC:
18A30
81S05
81T05
81T27

Keywords:
Quantum field theory
Second quantization
Fock representation
Regularization
Projective limits
Algebras of observables

ABSTRACT

In this series of papers, we investigate the projective framework initiated by Kijowski (1977) and Okołów (2009, 2014, 2013) [1,2], which describes the states of a quantum theory as projective families of density matrices. A short reading guide to the series can be found in Lanéry (2016).

A strategy to implement the dynamics in this formalism was presented in our first paper Lanéry and Thiemann (2017) (see also Lanéry, 2016, section 4), which we now test in two simple toy-models. The first one is a very basic linear model, meant as an illustration of the general procedure, and we will only discuss it at the classical level. In the second one, we reformulate the Schrödinger equation, treated as a classical field theory, within this projective framework, and proceed to its (non-relativistic) second quantization. We are then able to reproduce the physical content of the usual Fock quantization.

1. Introduction

In [3, section 3], we introduced a strategy to deal with dynamical constraints in a projective limit of symplectic manifolds. A regularization of these constraints will in general be necessary, since we cannot expect them to be adapted to the projective system, and we adopted the perspective that a dynamical state can be identified with the family of successive approximations approaching an exact solution of the dynamics. On the one hand, this allows us to put the dynamical state space into a projective form. On the other hand, it also provides a suitable ground for a notion of convergence, that will make it possible to define meaningful physical observables on this state space.

However, applying this procedure demands that one sets up a regularization scheme fulfilling a number of restrictive properties (summarized in [3, prop. 3.23]), which raises the question of its practicability. Hence, we now want to discuss two simple examples, meant as 'proofs of concept' that such schemes can indeed be designed.

Note that the framework in [3, section 3] was purely classical. We have not yet undertaken to formulate a general procedure regarding the resolution of dynamical constraints in projective systems of quantum state spaces [1,2,4–7]. Nevertheless, our second example will explore how analogous ideas can be implemented at the quantum level, and will give us the opportunity to delineate an appropriate course and to underline possible difficulties.

☆ Arxiv preprint number: 1411.3591.

* Correspondence to: Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, Campus Morelia, Apartado Postal 61-3 (Xangari), C.P. 58089, Morelia, Michoacán, Mexico.

E-mail address: suzanne.lanery@fau.de (S. Lanéry).

http://dx.doi.org/10.1016/j.geomphys.2017.08.007
0393-0440/

2. Linear constraints on a Kähler vector space

This first example is arguably mostly artificial and does not pretend to have great physical relevance. Our motivation here is to illustrate the concepts introduced in [3, sections 2 and 3] in the simplest possible setup. We consider an infinite dimensional Hilbert space $\mathcal{H}$ (which is nothing but a linear Kähler manifold) and form its rendering by a projective structure of finite dimensional Hilbert spaces (to prevent any confusion: the Hilbert spaces in discussion here are the phase spaces of classical systems, there will be nothing quantum in the present section). This rendering is built from an Hilbert basis of $\mathcal{H}$ by considering all the vector subspaces of $\mathcal{H}$ spanned by a finite number of basis vectors and linking them by orthogonal projections (a more satisfactory rendering for $\mathcal{H}$, namely one that does not require the choice of a preferred basis, will be presented in Section 3; however we do not want to use it here, since the constraints we will be looking at could be directly formulated as an elementary reduction over a cofinal part of its label set, and it would therefore not be appropriate as an example for the regularization procedure).

Proposition 2.1. *Let $\mathcal{H}, \langle \cdot\,, \cdot \rangle$ be a complex Hilbert space and define:*

1. $\forall v \in \mathcal{H}, J\, v := i\, v;$
2. $\forall v, w \in \mathcal{H}, \Omega(v,\, w) := 2\, \mathrm{Im}\, (\langle v,\, w\rangle).$

Then, $\mathcal{H},\ \Omega,\ J$ is a Kähler manifold.

Proof. The real scalar product Re $\langle \cdot\,, \cdot \rangle$ equips $\mathcal{H}$ (seen as a real vector space) with a structure of real Hilbert space, therefore, any bounded real-valued real-linear form on $\mathcal{H}$ can be written as $\mathrm{Re}\ \langle v,\ \cdot\rangle = 2\,\mathrm{Im}\left\langle -\frac{i}{2}\, v,\ \cdot\right\rangle = \Omega\left(-\frac{i}{2}\, v,\ \cdot\right)$ for some $v \in \mathcal{H}$. Hence, Ω is a strong symplectic structure.

Next, J is by construction a complex structure on $\mathcal{H}$. We have $\forall v, w \in \mathcal{H},\ \Omega(i\,v,\ i\,w) = \Omega(v,\ w)$, and $v \mapsto \Omega(v,\ i\,v) = 2\,\mathrm{Re}\,\langle v,\ v\rangle$ is positive definite.

The integrability conditions for Ω and J are trivially satisfied since we actually have a Kähler *vector space*. □

Proposition 2.2. *Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space (equipped with the strong symplectic structure Ω defined in Proposition 2.1) and let $(e_i)_{i\in\mathbb{N}}$ be an Hilbert basis of $\mathcal{H}$. We define:*

1. $\mathcal{L} := \{I \subset \mathbb{N} \mid 0 < \#I < \infty\}$ *equipped with the preorder defined by* $\subset$*;*
2. $\forall I \in \mathcal{L},\ \mathcal{H}_I := \mathrm{Vect}\,\{e_i \mid i \in I\}$ *equipped with the induced symplectic structure Ω_I (which is also the natural symplectic structure on $\mathcal{H}_I$ as a finite dimensional Hilbert space);*
3. $\forall I \subset I' \in \mathcal{L},\ \pi_{I'\to I} := \Pi_I|_{\mathcal{H}_{I'}\to\mathcal{H}_I}$ *where Π_I is the orthogonal projection on $\mathcal{H}_I$;*
4. $\mathcal{H}_{\mathbb{N}} := \mathcal{H}$ *and* $\forall I \in \mathcal{L},\ \pi_{\mathbb{N}\to I} := \Pi_I|_{\mathcal{H}\to\mathcal{H}_I}$.

Then, this defines a rendering [3, def. 2.6] of the symplectic manifold $\mathcal{H}$ by the projective system of phase spaces $(\mathcal{L},\ \mathcal{H},\ \pi)^{\downarrow}$. We define $\sigma_{\downarrow} : \mathcal{H} \to \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{H},\pi)}$ as in [3, def. 2.6].

Additionally, defining the dense vector subspace of $\mathcal{H}$, $\mathcal{D} := \mathrm{Vect}\,\{e_i \mid i \in \mathbb{N}\}$ (without completion, i.e. the space of finite linear combinations of the e_i), we have a bijective antilinear map $\zeta : \mathcal{D}^* \to \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{H},\pi)}$ such that $\zeta^{-1} \circ \sigma_{\downarrow} : \mathcal{H} \to \mathcal{D}^*$ is the canonical identification of $\mathcal{H}$ with $\mathcal{D}' \subset \mathcal{D}^*$ (where $\mathcal{D}^*$ is the algebraical dual of $\mathcal{D}$ and $\mathcal{D}'$ the topological one).

Proof. $\mathcal{L}$ is a directed set, since $\forall I, I' \in \mathcal{L},\ I \cup I' \in \mathcal{L}$ and $I, I' \subset I \cup I'$.

Let $I, I' \in \mathcal{L} \sqcup \{\mathbb{N}\}$ with $I \subset I'$. $\pi_{I'\to I}$ is surjective by construction. Next, since $\mathcal{H}_I$ is closed, we have, for any bounded real-valued real-linear form υ on $\mathcal{H}_I$, a vector $\underline{\upsilon} \in \mathcal{H}_I$ such that:

$$\forall v \in \mathcal{H}_I,\ \upsilon(v) = \Omega_I(\underline{\upsilon},\ v) = \mathrm{Re}\ \left\langle 2i\,\underline{\upsilon},\ v\right\rangle_I .$$

Hence, since Π_I is the $\mathbb{C}$-orthogonal projection on the complex vector subspace $\mathcal{H}_I$, it is also the $\mathbb{R}$-orthogonal projection on the real vector subspace $\mathcal{H}_I$, and we have:

$$\forall v \in \mathcal{H}_{I'},\ \upsilon \circ \pi_{I'\to I}(v) = \mathrm{Re}\ \left\langle 2i\,\underline{\upsilon},\ \Pi_I\, v\right\rangle_I = \mathrm{Re}\ \left\langle 2i\,\underline{\upsilon},\ v\right\rangle_{I'} = \Omega_{I'}(\underline{\upsilon},\ v),$$

and therefore $\pi_{I'\to I}\left(\underline{\upsilon \circ \pi_{I'\to I}}\right) = \pi_{I'\to I}\left(\underline{\upsilon}\right) = \underline{\upsilon}$.

Clearly for $I \in \mathcal{L}$, we have $\pi_{I\to I} = \mathrm{id}_{\mathcal{H}_I}$ and for $I, I', I'' \in \mathcal{L} \sqcup \{\mathbb{N}\}$ with $I \subset I' \subset I''$, $\pi_{I'\to I} \circ \pi_{I''\to I'} = \pi_{I''\to I}$.

Lastly, we define:

$$\begin{array}{cccc} \zeta & : & \mathcal{D}^* & \to & \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{H},\pi)} \\ & & \upsilon & \to & \left(\overline{\upsilon|_{\mathcal{H}_I}}\right)_{I\in\mathcal{L}} \end{array} ,$$

where for all $I \in \mathcal{L}$, $\overline{(\,\cdot\,)} : \mathcal{H}_I^* \to \mathcal{H}_I$ is the canonical identification provided by the complex Hilbert space structure on $\mathcal{H}_I$ ($\mathcal{H}_I$ is finite dimensional, hence $\mathcal{H}_I^* = \mathcal{H}_I'$).

The map ζ is well-defined, since $\forall I \subset I' \in \mathcal{L},\ \forall v \in \mathcal{H}_I,\ \left\langle \pi_{I'\to I}\left(\overline{\upsilon|_{\mathcal{H}_{I'}}}\right),\ v\right\rangle_I = \left\langle \overline{\upsilon|_{\mathcal{H}_{I'}}},\ v\right\rangle_{I'} = \upsilon(v) = \left\langle \overline{\upsilon|_{\mathcal{H}_I}},\ v\right\rangle_I$, hence $\pi_{I'\to I}\left(\overline{\upsilon|_{\mathcal{H}_{I'}}}\right) = \overline{\upsilon|_{\mathcal{H}_I}}$.