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The variational minimization solutions of 3-body problems with 2 fixed centers

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ABSTRACT

In the Newtonian 3-body problem with two fixed centers in R^3, two particles with equal masses are assumed fixed and the trajectory of the third particle is affected by the two fixed particles according to Newton's second law and the general gravitational law. We show that the minimization of the action functional on suitable classes of loops yields collision-free periodic orbits of the 3-body problem provided that some simple conditions on the period are satisfied; more precisely, geometric information is obtained for such variational minimization solutions.

1. Introduction and main results

The study of the 3-body problem with two fixed centers can be traced back to the eighteenth century with contributions by Euler [1–3]. We assume two particles with equal masses of value 1 are fixed at positions $q_1 = (1, 0, 0)$ and $q_2 = (-1, 0, 0)$, respectively, while the third particle with mass m is affected by the two fixed particles and moves according to Newton's second law (Newton's second law says that the mass times the acceleration of a particle is equal to the sum of the forces acting on the particle) and the universal gravitational law. This means that we shall look for a periodic solution to the equations of motion which describes the behavior of one body moving in the force field generated by two fixed particles. In this system, the position $q(t)$ for the third particle satisfies

$$m\ddot{q}(t) = \frac{m(q_1 - q(t))}{|q_1 - q(t)|^3} + \frac{m(q_2 - q(t))}{|q_2 - q(t)|^3}; \tag{1.1}$$

equivalently,

$$\ddot{q}(t) = \frac{(q_1 - q(t))}{|q_1 - q(t)|^3} + \frac{(q_2 - q(t))}{|q_2 - q(t)|^3}. \tag{1.2}$$

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In this paper, periodic solutions of systems (1.2) are found as critical points of the associated action integral

$$I(q)=\int_0^T\left[\frac{1}{2}|\dot{q}(t)|^2+\frac{1}{|q_1-q(t)|}+\frac{1}{|q_2-q(t)|}\right]dt \tag{1.3}$$

over a suitable function space.

The type of system (1.2) belongs to the class of Hamiltonian systems with singular potentials. In the last 30 years, variational methods have played a key role in the study of singular Hamiltonian systems by many researchers (see [1,4–19]). The main difficulty in using variational methods has been the elimination of collisions. Several techniques have been developed to avoid collisions for Newtonian N-body problems (see [1,4–14,20]); in particular, our current approach follows the one introduced in [10]. The results on the 3-body problem using variational methods in [15] cover the strong-force case in which collisions can be excluded in a standard way, but no geometric information for the periodic solutions in the case of two fixed centers was obtained for such variational minimization.

To overcome the lack of compactness, we define the action functional $I(q)$ on the loop space

$$W_0^{1,2}=\left\{q(t)\in W^{1,2}(\mathbb{R}/\mathbb{R}T,\mathbb{R}^3)\,\Big|\int_0^T q(t)dt=0\right\}, \tag{1.4}$$

where

$$W^{1,2}(\mathbb{R}/\mathbb{R}T,\mathbb{R}^3)=\{x(t)|x(t)\text{ is absolutely continuous}, \tag{1.5}$$

$$x(t+T)=x(t),\ \dot{x}(t)\in L^2(\mathbb{R},\mathbb{R}^3)\}. \tag{1.6}$$

The existence of periodic solutions can be derived from the minimization of the action integral. To avoid collisions, let

$$\Lambda=\{q(t)\in W_0^{1,2}|q(t)\neq q_1\text{ and }q(t)\neq q_2\} \tag{1.7}$$

and $\overline{\Lambda}$ the weakly closed set of Λ.

Based on the method of Long and Zhang in [10], we shall establish following theorem.

Theorem 1.1. *The functional $I(q)$ defined by (1.3) possesses a positive minimum on $\overline{\Lambda}$ and the minimizers $Q(t)$ are collision-free periodic orbits; furthermore, we have:*
(1) $Q(t)$ is a nontrivial circular periodic solution centered at the origin in yoz-plane, when $T>\sqrt{2}\pi$;
(2) $Q(t)$ is a trivial solution; that is, $Q(t)=(0,0,0)$ when $T\leq\sqrt{2}\pi$.

2. The proof of Theorem 1.1

We record the following inequalities which will be needed.

Lemma 2.1 (*Poincaré–Wirtinger Inequality*). *If $q\in W^{1,2}(\mathbb{R}/\mathbb{Z}\mathbb{T},\mathbb{R}^d)$ and $\int_0^T q(t)dt=0$, then $\int_0^T|\dot{q}(t)|^2dt\geq(\frac{2\pi}{T})^2\int_0^T|q(t)|^2dt$. Equality is achieved if and only if $q(t)=\alpha\cos\frac{2\pi}{T}t+\beta\sin\frac{2\pi}{T}t$, $\alpha,\beta\in\mathbb{R}^d$.*

Lemma 2.2 (*Jensen's Inequality [21]*). *Suppose that $\alpha\leq f(x)\leq\beta$, where α and β may be finite or infinite, the range of integration and the weight function $p(x)$ is finite and positive everywhere, and $\phi''(t)$ is positive finite for $\alpha<t<\beta$. Then*

$$\phi\left(\frac{\int_\alpha^\beta f(x)p(x)dx}{\int_\alpha^\beta p(x)dx}\right)\leq\left(\frac{\int_\alpha^\beta\phi(f(x))p(x)dx}{\int_\alpha^\beta p(x)dx}\right), \tag{2.1}$$

whenever the right-hand side exists and is finite. Equality occurs only when $f(x)\equiv C$.

As direct consequences of Lemma 2.1 in [6] and Proposition 2.2 in [10], we have the following lemma.

Lemma 2.3. *If $q_1+q_2=(0,0,0)$, then*

$$\sum_{i=1}^{2}\frac{1}{|q_i-q|}\geq 2^{3/2}(2|q|^2+|q_1|^2+|q_2|^2)^{-1/2}. \tag{2.2}$$

Equality holds in (2.2) if and only if $|q_1-q|=|q_2-q|$.

For the reader's convenience, a proof of Lemma 2.3 is given.