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A stellar model with diffusion in general relativity

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ABSTRACT

We consider a spherically symmetric stellar model in general relativity whose interior consists of a pressureless fluid undergoing microscopic velocity diffusion in a cosmological scalar field. We show that the diffusion dynamics compel the interior to be spatially homogeneous, by which one can infer immediately that within our model, and in contrast to the diffusion-free case, no naked singularities can form in the gravitational collapse. We then study the problem of matching an exterior Bondi type metric to the surface of the star and find that the exterior can be chosen to be a modified Vaidya metric with variable cosmological constant. Finally, we study in detail the causal structure of an explicit, self-similar solution.

1. Introduction

The structure of singularities formed in the gravitational collapse of bounded matter distributions is a widely investigated problem in general relativity. The question of whether such singularities are naked, i.e., visible to far-away observers, or whether on the contrary they are safely hidden inside a black hole, has been the subject of innumerable works in the physical and mathematical literature. Nevertheless the problem remains poorly understood, even in the idealized case in which the collapsing body is spherically symmetric. A complete solution is only available for the gravitational collapse of a spherically symmetric dust cloud and can be found in the pioneering works by Oppenheimer–Snyder [1] and Christodoulou [2]. (In [3,4] Christodoulou analyzes the question of existence and stability of naked singularities in the gravitational collapse of a massless scalar field. Notwithstanding the importance of these works, our focus is on matter models that describe material bodies, such as perfect fluids and kinetic particles [5].)

The Oppenheimer–Snyder model consists of a collapsing spatially homogeneous and isotropic dust interior, described by the contracting Friedmann–Lemaître solution, matched at a comoving boundary with a Schwarzschild exterior. Dropping the homogeneity assumption of the interior leads to the class of Lemaître–Tolman–Bondi solutions. These inhomogeneous stellar models were studied numerically by Eardley and Smarr [6], and analytically by Christodoulou [2]. It was shown that, in contrast to the case studied by Oppenheimer and Snyder, the spatially inhomogeneous collapse leads to the formation of naked singularities. See the prolog of [7] for a historical review on the gravitational collapse problem in general relativity.

In this paper we initiate the study of the gravitational collapse of matter subject to diffusion. We believe that the inclusion of diffusion dynamics in the gravitational collapse problem is meaningful both from a mathematical and physical point of view. From one hand it is well known that diffusion terms introduce a regularizing effect in the equations, which might prevent the formation of naked singularities in general relativity. On the other hand the physical relevance of diffusion phenomena is unquestionable and the applications in general relativity have been discussed in [8–10].

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We begin our study with the simplest possible model, namely a spherically symmetric dust cloud undergoing diffusion in a cosmological scalar field. In this case the regularizing effect due to diffusion is overwhelming: The interior of the dust cloud is forced to be spatially homogeneous. By this fact one can easily infer that, in contrast to the diffusion-free scenario described above, naked singularities cannot form in the gravitational collapse of a spherically symmetric dust cloud in the presence of diffusion.

Another interesting property of our model is that, in contrast to the diffusion-free case [1,2,11], the exterior of the star cannot be static. The simplest spherically symmetric solution of the Einstein equations that can provide a suitable exterior region for our stellar model is given by a Vaidya type metric which includes a variable cosmological constant (a generalization of the radiating version of the Schwarzschild-(Anti-)de-Sitter family of solutions).

A detailed analysis of our model is given in the following sections. We conclude this introduction by outlining the diffusion theory of matter in general relativity. There exist two versions of this theory: a kinetic one [8], which is based on a Fokker–Planck equation for the particle density in phase-space, and a fluid one [9], which is the (formal) macroscopic limit of the kinetic theory. In the present paper we apply the fluid theory. We recall that the energy–momentum tensor and current density of a perfect fluid are

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu), \qquad J^\mu = n u^\mu, \tag{1}$$

where ρ is the rest-frame energy density, p the pressure, u the four-velocity and n the particle density of the fluid. The diffusion behavior is imposed by postulating the equations

$$\nabla_\mu T^{\mu\nu} = \sigma J^\nu, \tag{2a}$$

$$\nabla_\mu (n u^\mu) = 0, \tag{2b}$$

where $\sigma > 0$ is the diffusion constant, which measures the energy gained by the particles per unit of time due to the action of the diffusion forces. The second equation entails the conservation of the total number of fluid particles.

By projecting (2a) along the direction of u^μ and onto the hypersurface orthogonal to u^μ, we obtain the following equations on the matter fields:

$$\nabla_\mu(\rho u^\mu) + p\nabla_\mu u^\mu = \sigma n, \tag{3a}$$

$$(\rho + p)u^\mu \nabla_\mu u^\nu + u^\nu u^\mu \nabla_\mu p + \nabla^\nu p = 0. \tag{3b}$$

The system (3) on the matter variables must be completed by assigning an equation of state between the pressure, the energy density and the particles number density. In this paper we assume that the fluid is pressureless (dust fluid). As the energy–momentum tensor of the fluid is not divergence-free, see (2a), we have to postulate the existence of an additional matter field in spacetime to re-establish the (local) conservation of energy. The role of this additional matter field is that of the solvent matter in which the diffusion of the fluid particles takes place. The simplest choice is to assume the existence of a vacuum energy scalar field ϕ with energy–momentum tensor $-\phi g_{\mu\nu}$, which leads to the following Einstein equations for the spacetime metric g (in units $8\pi G = c = 1$):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \phi g_{\mu\nu} = T_{\mu\nu}. \tag{4}$$

The evolution equation on the scalar field ϕ determined by (4), the Bianchi identities, and the diffusion equation (2a) is

$$\nabla_\nu \phi = \sigma J_\nu. \tag{5}$$

2. The stellar model

Throughout the rest of the paper we assume that spacetime (M, g) is spherically symmetric. The particle number density $n : M \to [0, \infty)$ and the four-velocity $u_x : T_x M \to \mathbb{R}^4$ at each point $x \in M$ are also spherically symmetric and satisfy (2b). Under these assumptions one can cover an open neighborhood U of the center of symmetry by a coordinate system (t, R, θ, ψ) such that the metric and the four-velocity take the form

$$g = -e^{2\Phi} dt^2 + e^{2\Psi} dR^2 + r^2 d\Omega^2 \quad u = e^{-\Phi}\partial_t \quad \text{in } U.$$

Here Φ, Ψ, r are functions of (t, R) and $d\Omega^2 = d\theta^2 + \sin^2\theta d\psi^2$ is the standard metric on S^2. The center of symmetry is defined by the timelike curve $r(t, R) = 0$. This coordinate system is called *comoving* and it is defined up to a transformation $t \to F(t), R \to G(R)$ of the time and radial coordinates.

2.1. The interior

We assume that the interior of the star defines a region V of spacetime covered by comoving coordinates. In particular, we assume that there exist $T \in (0, +\infty]$ and $R_b > 0$ such that $0 \le t < T$ and $0 \le R < R_b$ within $V \subset U$. The timelike hypersurface given by

$$\Sigma \; : \; R = R_b,$$