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About the geometry of the Earth geodetic reference surfaces

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A B S T R A C T

The paper focuses on the comparison of metrics of three most common reference surfaces of the Earth used in geodesy (excluding the plane which also belongs to reference surfaces used in geodesy when dealing with small areas): a sphere, an ellipsoid of revolution and a triaxial ellipsoid. The two latter surfaces are treated in a more detailed way. First, the mathematical form of the metric tensors using three types of coordinates is derived and the lengths of meridian and parallel arcs between the two types of ellipsoids are compared. Three kinds of parallels, according to the type of latitude, can be defined on a triaxial ellipsoid. We show that two types of parallels are spatial curves and one is represented by ellipses. The differences of curvature of both kinds of ellipsoid are analysed using the normal curvature radii. Priority of the chosen triaxial ellipsoid is documented by its better fit with respect to the high-degree geoid model EIGEN6c4 computed up to degree and order 2160.

0. Introduction

Geodesy uses several reference surfaces which approximate the physical surface of the Earth for solving problems. These surfaces generalize its true form by suitable approximation while the level of approximation depends on the nature of the problem. Such surfaces are a sphere, ellipsoid of revolution, triaxial ellipsoid, spheroid and geoid. This paper deals with the basic geometric aspects of these surfaces focusing on differences in their metrics expressed by the metric tensors in various coordinates. It focuses mainly on differences between the metrics of ellipsoid of revolution and triaxial ellipsoid. The historical overview of triaxial ellipsoid as a reference surface of the Earth is well described in [1]. The basic equations concerning on this topic were published by Burša and Pěč [2] and Feltens [3]. This paper extends and further develops their work in several aspects. From the practical point of view the paper analyses several quantitative metrics and curvature indicators and also qualitative differences between the shapes of the coordinate lines, changes in geographic grid orthogonality or differences between the geographic coordinates defined on particular surfaces. Based on the obtained results the applications in which it would be useful to replace the ellipsoid of revolution by triaxial ellipsoid are suggested in Conclusion. The possible fields of application are geodesy in general (cadastre, mapping, engineering surveying, cartography), global geophysics, satellite geodesy or planetary sciences. The lack of symmetry of the triaxial ellipsoid is still a strong reason to stick with the biaxial ellipsoid which is mathematically more simple and easier to use. On the other hand, with the progress of computational tools this problem can be manageable.

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1. Metric tensors of Earth's reference surfaces

The metric tensor is a quantity that carries the basic information about the metric characteristics of a particular surface. On a surface we can express a point location using a pair of general curvilinear coordinates u^α ($\alpha = 1, 2$) and also a triplet spatial Cartesian coordinates x_i ($i = 1, 2, 3$). Conventionally, indices in 2D space are designated using the small case Greek alphabet while in 3D space by the Latin alphabet. On each reference surface we use the specific pair of geographic coordinates, longitude and latitude u^α ($\alpha = 1, 2$), u^1 – longitude and u^2 – latitude to establish a right-handed system of the basis vectors on a surface $\mathbf{b}_1, \mathbf{b}_2$ or $\mathbf{b}^1, \mathbf{b}^2$ respectively. We distinguish the spherical coordinates $u^1 = \Lambda_s, u^2 = \Phi_s$, the geodetic coordinates $u^1 = \lambda, u^2 = \varphi$, the geocentric coordinates $u^1 = \lambda, u^2 = \Phi_c$ and the reduced coordinates $u^1 = \lambda_r, u^2 = \beta$. The Einstein summation convention is used throughout the paper.

1.1. Sphere

The general equation of the sphere $f_s = f_s(\Lambda_s, \Phi_s)$ with the radius R can be written as follows

$$f_s: \quad x_1^2 + x_2^2 + x_3^2 = R^2, \quad \begin{aligned} x_1 &= R\cos\Lambda_s\cos\Phi_s, & R &= \sqrt{x_1^2 + x_2^2 + x_3^2}, \\ x_2 &= R\sin\Lambda_s\cos\Phi_s, & \Lambda_s &= \arctan\left(\frac{x_2}{x_1}\right), \\ x_3 &= R\sin\Phi_s, & \Phi_s &= \arctan\left(\frac{x_3}{\sqrt{x_1^2+x_2^2}}\right). \end{aligned} \tag{1}$$

Relations (1) between the Cartesian coordinates x_i and curvilinear coordinates Λ_s, Φ_s determine the form of the covariant basis $\mathbf{b}_\alpha$ and the contravariant basis $\mathbf{b}^\beta$ [4, p. 135]

$$\mathbf{b}_1 = \begin{pmatrix} \dfrac{\partial x_1}{\partial \Lambda_s} \\ \dfrac{\partial x_2}{\partial \Lambda_s} \\ \dfrac{\partial x_3}{\partial \Lambda_s} \end{pmatrix} = \begin{pmatrix} -R\sin\Lambda_s\cos\Phi_s \\ R\cos\Lambda_s\cos\Phi_s \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} \dfrac{\partial x_1}{\partial \Phi_s} \\ \dfrac{\partial x_2}{\partial \Phi_s} \\ \dfrac{\partial x_3}{\partial \Phi_s} \end{pmatrix} = \begin{pmatrix} -R\cos\Lambda_s\sin\Phi_s \\ -R\sin\Lambda_s\sin\Phi_s \\ R\cos\Phi_s \end{pmatrix}, \tag{2a}$$

$$\mathbf{b}^1 = \begin{pmatrix} \dfrac{\partial \Lambda_s}{\partial x_1} \\ \dfrac{\partial \Lambda_s}{\partial x_2} \\ \dfrac{\partial \Lambda_s}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \dfrac{-x_2}{x_1^2+x_2^2} \\ \dfrac{x_1}{x_1^2+x_2^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{-\sin\Lambda_s}{R\cos\Phi_s} \\ \dfrac{\cos\Lambda_s}{R\cos\Phi_s} \\ 0 \end{pmatrix}, \quad \mathbf{b}^2 = \begin{pmatrix} \dfrac{\partial \Phi_s}{\partial x_1} \\ \dfrac{\partial \Phi_s}{\partial x_2} \\ \dfrac{\partial \Phi_s}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \dfrac{-x_1x_3}{R^2\sqrt{x_1^2+x_2^2}} \\ \dfrac{-x_2x_3}{R^2\sqrt{x_1^2+x_2^2}} \\ \dfrac{\sqrt{x_1^2+x_2^2}}{R^2} \end{pmatrix} = \begin{pmatrix} \dfrac{-\cos\Lambda_s\sin\Phi_s}{R} \\ \dfrac{-\sin\Lambda_s\sin\Phi_s}{R} \\ \dfrac{\cos\Phi_s}{R} \end{pmatrix}. \tag{2b}$$

The basis vectors $\mathbf{b}_\alpha$, by definition (2a), are tangential to the coordinate lines on a sphere. The basis vectors $\mathbf{b}^\beta$, from definition (2b), are perpendicular to the coordinate surfaces. Components of the covariant and contravariant metric tensors of a surface $\mathbf{g}_{\alpha\beta}$ and $\mathbf{g}^{\alpha\beta}$ respectively, are defined as inner products of the basis vectors, see Spiegel [4], so that on a sphere after applying Eqs. (2a) and (2b) hold

$$\mathbf{g}_{\alpha\beta} = \mathbf{b}_\alpha \cdot \mathbf{b}_\beta = \begin{pmatrix} R^2\cos^2\Phi_s & 0 \\ 0 & R^2 \end{pmatrix}, \quad \mathbf{g}^{\alpha\beta} = \mathbf{b}^\alpha \cdot \mathbf{b}^\beta = \begin{pmatrix} \dfrac{1}{R^2\cos^2\Phi_s} & 0 \\ 0 & \dfrac{1}{R^2} \end{pmatrix}. \tag{3}$$

From Eq. (3) one can see that $\mathbf{g}_{\alpha\beta}\mathbf{g}^{\alpha\beta} = \mathbf{E} \Rightarrow \mathbf{g}_{\alpha\beta} = (\mathbf{g}^{\alpha\beta})^{-1}$, where $\mathbf{E}$ is a unit matrix. This results from the property of the conjugate bases $\mathbf{b}_\alpha, \mathbf{b}^\beta$, which fulfil the reciprocity conditions $\mathbf{b}_\alpha \cdot \mathbf{b}^\beta = \delta_\alpha^\beta$, where δ_α^β is the Kronecker delta.

1.2. Ellipsoid of revolution

On an ellipsoid of revolution we use three types of curvilinear coordinates: geodetic coordinates $u^1 = \lambda,\ u^2 = \varphi$, reduced coordinates $u^1 = \lambda,\ u^2 = \beta$ and geocentric coordinates $u^1 = \lambda,\ u^2 = \Phi_c$. We demonstrate the advantage of using the