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Classification and equivariant cohomology of circle actions on 3d manifolds

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ABSTRACT

The classification of Seifert manifolds was given in terms of numeric data by Seifert (1933), and then generalized by Raymond (1968) and Orlik and Raymond (1968) to circle actions on closed 3d manifolds. In this paper, we further generalize the classification to circle actions on 3d manifolds with boundaries by adding a numeric parameter and a graph of cycles. Then, we describe the rational equivariant cohomology of 3d manifolds with circle actions in terms of ring, module and vector-space structures. We also compute equivariant Betti numbers and Poincaré series for these manifolds and discuss the equivariant formality.

1. Introduction

The classification of closed 3d manifolds with "nice" decompositions into circles was given by Seifert [1] in terms of principal Euler number b, orientability ϵ and genus g of the underlying 2d orbifolds, and pairs of coprime integers (m_i, n_i) called Seifert invariants. Hence, these manifolds were given the name Seifert manifolds.

Later, the classification was generalized by Orlik and Raymond [2,3] to circle actions on closed 3d manifolds allowing fixed points and special exceptional orbits. Orlik and Raymond found that in their case the underlying 2d orbifolds have circle boundaries contributed by the fixed points and special exceptional orbits. Hence, besides the four types of numeric data used by Seifert, two more types of numeric data were introduced by Orlik and Raymond: the number f of fixed components and the number s of special exceptional components. Then, Orlik and Raymond proved the following:

Theorem (*Orlik-Raymond Classification of closed 3d S^1-manifolds, [2,3]*). *Let S^1 act effectively and smoothly on a closed, connected smooth 3d manifold M. Then, the orbit invariants*

$$\left\{b; (\epsilon, g, f, s); (m_1,\ n_1), \ldots, (m_l,\ n_l)\right\}$$

determine M up to equivariant diffeomorphisms, subject to certain conditions. Conversely, any such set of invariants can be realized as a closed 3d manifold with an effective S^1-action.

The first goal of this paper is to further generalize the Orlik–Raymond Classification Theorem to circle actions on compact 3d manifolds, allowing boundaries. By the classification of circle actions on closed 2d manifolds, those boundaries have to be tori $\mathbb{T}$, spheres S^2, projective planes $\mathbb{R}P^2$ or Klein bottles K. Our approach relies on a careful discussion on the equivariant

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neighbourhoods of non-principal orbits and boundaries. Let t be the number of torus boundaries and $\mathcal{G}$ be a graph of cycles to keep track of the boundary types $S^2,\ K,\ \mathbb{R}P^2$, we get the following:

Theorem 3.1. *Let the circle group S^1 act effectively and smoothly on a compact, connected 3d manifold M, possibly with boundary. Then, the orbit invariants*

$$\left\{b; (\epsilon, g, f, s, t); (m_1,\ n_1), \ldots, (m_l,\ n_l); \mathcal{G}\right\}$$

consisting of numeric data and a graph of cycles, determine M up to equivariant diffeomorphisms, subject to certain conditions. Conversely, any such set of invariants can be realized as a 3d manifold with an effective S^1-action.

Using the Orlik–Raymond Theorem, one can compute the fundamental groups, ordinary homology and cohomology with $\mathbb{Z}$ or $\mathbb{Z}_p$ coefficients for closed 3d S^1-manifolds, (cf.[4–7]). In this paper, we are instead interested in equivariant topological invariants.

Hence, the second goal of this paper is to describe the $\mathbb{Q}$-coefficient equivariant cohomology of compact 3d manifold M with circle action. Our main strategy is to apply the equivariant Mayer–Vietoris sequence to a decomposition of the manifold M into a fixed-point-free part and a neighbourhood of the fixed-point set. Then, we get

Theorem 4.2. *Let M be a compact connected 3d manifold(possibly with boundary) with an effective S^1-action, and F be its fixed-point set(possibly empty), then there is a short exact sequence of cohomology groups in $\mathbb{Q}$ coefficients:*

$$0 \to H^*_{S^1}(M) \to H^*(M/S^1) \oplus \Big(\mathbb{Q}[u] \otimes H^*(F)\Big) \to H^*(F) \to 0$$

Using this theorem, we can describe the ring, module and vector-space structures of the equivariant cohomology $H^*_{S^1}(M)$ in details. Furthermore, we will calculate equivariant Betti numbers and Poincaré series, and discuss a numeric condition for equivariant formality.

2. S^1-actions on 2d manifolds and closed 3d manifolds

In this section, we will recall the classification of effective S^1-actions on closed manifolds in dimensions 2 and 3, which will be crucial for our classification of effective S^1-actions on 3d manifolds with boundaries. All these results are well known, and can be found in greater details from the original papers by Orlik and Raymond [2,3] or the notes and books [4,8–10].

2.1. Some basic facts about group actions on manifolds

Throughout the paper, we always assume that a manifold M is compact, smooth and connected, and a group G is compact, unless otherwise mentioned. For convenience, we will denote a G-action on M as $G \curvearrowright M$. The quotient M/G is called the **orbit space** of the G-action on M. For any point x in M, let $G_x = \{g \in G \mid g \cdot x = x\}$ be its stabilizer. We write $M^G = \{x \in M \mid G_x = G\}$ for the set of fixed points. If $G_x = G$ for every $x \in M$, we say that the G-action on M is **trivial**. If $G_x = \{1\}$ for every $x \in M$, we say that the G-action on M is **free**. If the intersection $\cap_{x\in M} G_x = \{1\}$, we say that the G-action on M is **effective**. Throughout this paper, group actions are usually assumed to be effective, unless otherwise mentioned.

For any orbit $G \cdot x$, let V_x be an orthogonal complement of $T_x(G \cdot x)$ in T_xM. The infinitesimal action of G_x on T_xM gives a linear **isotropy representation** $G_x \curvearrowright V_x$. Then, the normal bundle of the orbit $G \cdot x$ can be written as

$$G\times_{G_x} V_x = \left\{[g, v] \mid (g, v) \sim (gh, h^{-1}v) \text{ for any } h \in G\right\}$$

with a G-action induced from the canonical G-action on the left of the first factor of $G \times V_x$.

The following theorem, proved by Koszul [11], equivariantly identifies the normal bundle with the tubular neighbourhood of an orbit $G \cdot x$.

Theorem 2.1 (*The slice theorem, [11]*)**.** *There exists an equivariant exponential map*

$$\exp : G\times_{G_x} V \longrightarrow M$$

which is an equivariant diffeomorphism from an open neighbourhood of the zero section $G\times_{G_x}\{0\}$ in $G\times_{G_x} V_x$ to an equivariant neighbourhood of $G \cdot x$ in M.

Thus, an equivariant neighbourhood of the orbit $G \cdot x$ can be specified in terms of the stabilizer G_x and the isotropy representation of G_x on the normal vector space.

Similar to the ordinary non-equivariant case, the equivariant identification between normal bundles and neighbourhoods generalizes beyond single orbit to submanifold and boundary, cf. Kankaanrinta [12].