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Type one generalized Calabi–Yaus

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ABSTRACT

We study type one generalized complex and generalized Calabi–Yau manifolds. We introduce a cohomology class that obstructs the existence of a globally defined, closed 2-form which agrees with the symplectic form on the leaves of the generalized complex structure, the *twisting class*. We prove that in a compact, type one, $4n$-dimensional generalized complex manifold the Euler characteristic must be even and equal to the signature modulo four. The generalized Calabi–Yau condition places much stronger constraints: a compact type one generalized Calabi–Yau fibers over the 2-torus and if the structure has one compact leaf, then this fibration can be chosen to be the fibration by the symplectic leaves of the generalized complex structure. If the twisting class vanishes, one can always deform the structure so that it has a compact leaf. Finally we prove that every symplectic fibration over the 2-torus admits a type one generalized Calabi–Yau structure.

0. Introduction

Generalized complex structures, introduced by Hitchin in 2003 [1] and Gualtieri [2], are a simultaneous generalization of complex and symplectic structures. At each point p in the manifold M, a generalized complex structure is equivalent to a symplectic subspace of T_pM together with a transverse complex structure. The *type* of the generalized complex structure at the point in question is the complex dimension of this transverse complex space. Thus, symplectic structures have type 0 everywhere and complex structures have maximal type everywhere. In general, generalized complex structures determine singular symplectic foliations with a transverse complex structure and, if the type is constant, they determine ordinary symplectic foliations.

The type of a generalized complex structure on M is an integer-valued lower semi-continuous function with locally constant parity. In the generic even case, generalized complex structures may be viewed as symplectic structures with singularities along loci where the type jumps from 0 to 2 or higher; when these singular loci are required to satisfy a transversality condition, we have *stable generalized complex structures*, which were studied by Cavalcanti and Gualtieri in [3]. If, instead, we are in the case of odd type, a generic generalized complex structure would be of type 1 almost everywhere; very little is currently known about type 1 structures.

Besides being the odd analogue of symplectic structures, type 1 structures are closely related to stable (even) generalized complex structures in a more direct way: the singular locus of the symplectic form in the stable case inherits a type 1 *generalized Calabi–Yau structure*. So, the study of stable structures naturally leads to the study of type 1 generalized Calabi–Yau structures. The objective of this paper is to determine the basic differential topological properties of these structures.

We prove that a compact, connected, type 1 generalized Calabi–Yau manifold, M, has a rather restricted topology: M must be a fiber bundle over the 2-torus. Further, if the manifold has at least one compact symplectic leaf, then all leaves are

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compact symplectic manifolds, and M fibers over its symplectic leaf space, which is T^2. We also prove a partial converse to this statement: every compact symplectic fibration over T^2 admits a generalized Calabi–Yau structure for which the symplectic leaves are the fibers of the fibration. As a special case, we obtain a correspondence in four dimensions: a compact four-manifold admits a type 1 generalized Calabi–Yau structure if and only if it is an oriented fibration over T^2. These results may be viewed as the generalized complex analogues of the results obtained by Guillemin, Miranda and Pires for codimension one Poisson structures [4].

1. Topology of type one generalized complex structures

1.1. The twisting class of a generalized complex structure

Definition 1.1. A *generalized complex structure* on a manifold M with closed 3-form H is a complex structure $\mathbb{J}$ on $\mathbb{T}M = TM \oplus T^*M$ compatible with the natural pairing of vectors on forms and integrable with respect to the Courant bracket twisted by H.

Alternatively, $\mathbb{J}$ is fully determined by its $+i$-eigenspace $L \subset \mathbb{T}_{\mathbb{C}}M$, a maximal isotropic, involutive sub-bundle satisfying $L \cap \overline{L} = \{0\}$. Furthermore, $\mathbb{J}$ can be fully described using differential forms:

Definition 1.2. A *generalized complex structure* on a manifold with closed 3-form (M^{2n}, H), is a complex line bundle $K \subset \wedge^\bullet T^*_{\mathbb{C}}M$ such that

1. K is generated pointwise by a form ρ of the following algebraic type

$$\rho = e^{B+i\omega} \wedge \Omega,$$

where Ω is a decomposable form and B and ω are real two-forms;
2. Pointwise, for the generator above,

$$\Omega \wedge \bar{\Omega} \wedge \omega^{n-k} \neq 0,$$

where k is the degree of Ω;
3. For every nonvanishing local section ρ there is $X + \xi \in \Gamma(\overline{L})$ such that

$$d^H\rho := d\rho + H \wedge \rho = \iota_X\rho + \xi \wedge \rho.$$

The degree of the form Ω at a point p is the *type* of the generalized complex structure at p, the line bundle K is the *canonical bundle* and $X + \xi$ is the *modular field* corresponding to the trivialization ρ.

Definition 1.3. A *generalized Calabi–Yau structure* on (M, H) is a generalized complex structure determined by a nowhere vanishing d^H-closed form.

Examples of generalized complex manifolds include symplectic manifolds, (M, ω), where K is the line generated by $e^{i\omega}$; complex manifolds, where $K = \wedge^{n,0}T^*M$ is the usual canonical bundle; and holomorphic Poisson manifolds (M, I, π) where $K = e^{\pi} \cdot \wedge^{n,0}T^*M$ and π acts on forms by interior product.

From the definition we see that, pointwise, the subspace $\mathcal{D}$ annihilating $\Omega \wedge \bar{\Omega}$ is the complexification of a real subspace of TM and ω is a symplectic structure on $\mathcal{D}$. If the type is constant, $\mathcal{D}$ is an integrable distribution and ω is a symplectic structure on the fibers. In general, $\mathcal{D}$ is an integrable singular distribution and ω gives a symplectic structure to its leaves.

Any generalized complex structure $\mathbb{J}$ on M^{2n} decomposes the space of forms into its ik-eigenspaces: U^k. These are nontrivial for all the integers k between $-n$ and n with $U^n = K$ and $U^{n-k} = \wedge^k\overline{L} \cdot K$. Further the operator d^H also decomposes as a sum $d^H = \partial + \overline{\partial}$ with

$$\partial : \mathcal{U}^k \to \mathcal{U}^{k+1} \quad \text{and} \quad \overline{\partial} : \mathcal{U}^k \to \mathcal{U}^{k-1}.$$

Since L is involutive, it is a Lie algebroid over M and using the nondegenerate pairing to identify $\overline{L} = L^*$, $\Gamma(\wedge^\bullet\overline{L})$ becomes a differential graded Lie algebra (DGLA) with d_L, the Lie algebroid differential from L, and the Schouten–Nijenhuis extension of the Courant bracket as a bracket. Further the space of forms with the operator $\overline{\partial}$ is a differential module for this DGLA, i.e., for all $\rho \in \Omega^\bullet(M; \mathbb{C})$ and $\alpha \in \Gamma(\wedge^\bullet\overline{L})$ we have

$$\{\overline{\partial}, \alpha\}\rho = (d_L\alpha)\rho, \tag{1.1}$$

where $\{\cdot, \cdot\}$ is the graded commutator of linear differential operators on forms.

It follows directly from (1.1) that any (local) modular field is d_L-closed. If the canonical bundle is trivial, a global nowhere vanishing section $\rho \in \Gamma(K)$ gives rise to a global modular field v. Another nonvanishing section is just a multiple of ρ, say $g\rho$, where $g : M \to \mathbb{C}^*$, and hence the modular field of this new section is given by $v + d_L \log g$ and, if K is trivial, the cohomology class of the modular field in the complex

$$0 \longrightarrow \Gamma(M; \mathbb{C}^*) \xrightarrow{d_L \circ \log} \Gamma(M; \overline{L}) \xrightarrow{d_L} \Gamma(M; \wedge^2\overline{L}) \xrightarrow{d_L} \cdots$$