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Scalar curvature in conformal geometry of Connes–Landi noncommutative manifolds

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ABSTRACT

We first propose a conformal geometry for Connes–Landi noncommutative manifolds and study the associated scalar curvature. The new scalar curvature contains its Riemannian counterpart as the commutative limit. Similar to the results on noncommutative two tori, the quantum part of the curvature consists of actions of the modular derivation through two local curvature functions. Explicit expressions for those functions are obtained for all even dimensions (greater than two). In dimension four, the one variable function shows striking similarity to the analytic functions of the characteristic classes appeared in the Atiyah–Singer local index formula, namely, it is roughly a product of the j-function (which defines the $\hat{A}$-class of a manifold) and an exponential function (which defines the Chern character of a bundle). By performing two different computations for the variation of the Einstein–Hilbert action, we obtain deep internal relations between two local curvature functions. Straightforward verification for those relations gives a strong conceptual confirmation for the whole computational machinery we have developed so far, especially the Mathematica code hidden behind the paper.

1. Introduction

The general question behind this paper is to explore the notion of intrinsic curvature, which lies in the core of the geometry, in the operator theoretical framework (noncommutative differential geometry). The question remained intangible until the recent development of modular geometry on noncommutative two tori [1], other major references on this subject include [2–6], see also [7,8]. The computation has been extended to noncommutative four tori smoothly [9,10]. The essential computational tool is Connes' pseudo-differential calculus for C^*-dynamical system, first constructed in [11] and extended in [6] to Heisenberg modules. Some different approaches can be found in [12,13].

In the modular geometry on noncommutative tori, the Riemannian aspect is somehow hidden in the sense that the original metric is flat. To obtain a stronger demonstration that our approach does include Riemannian geometry as a special case, we would like to test the ideas on a larger class of deformed Riemannian manifolds known as Connes–Landi noncommutative manifolds, first introduced in [14]. The computation was initiated in the author's previous work [15]. The first main result in [15] is a pseudo differential calculus which is suitable for studying the spectral geometry. Such a calculus not only records the Riemannian curvature information but also dramatically simplify the computation. By testing the calculus on the scalar Laplacian operator Δ_φ in [1], we recovered the local curvature functions. This paper starts with an unfinished problem in [15], namely, computing the full scalar curvature for all even dimensional Connes–Landi noncommutative manifolds.

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Table 1
Conformal change of metric and the associated scalar curvature in the noncommutative setting.

Riemannian geometry	Connes–Landi spectral triples
e^{-2h} with $h \in C^\infty(M)$ real-valued	e^{-2h} with $h = h^* \in C^\infty(M_\theta)$ self-adjoint
Original metric g	Spinor Dirac $\not{D}$
$g' = e^{-2h}g$	$D_h = e^{h/2}\not{D}e^{h/2}$
Scalar curvature for g': $\mathcal{S}_{g'}$	$R_{D_h} \in C^\infty(M_\theta)$: local expression of $V_2(\cdot, D_h^2)$ defined in (1.2)

Let us first recall some basic notions in conformal geometry in the operator theoretical setting. The generalization to the noncommutative setting is straightforward, which leads to the conformal geometry of Connes–Landi noncommutative manifolds defined in Table 1.

For a closed Riemannian manifold (M, g), which is also spin with the spinor bundle $\not{S}_g$ and the spinor Dirac operator $\not{D}$, the spin geometry can be recovered by the Dirac model, which consists of the spectral data $(C^\infty(M), L^2(\not{S}_g), \not{D})$. This motivates the basic paradigm of noncommutative geometry: the notion of spectral triples $(\mathcal{A}, \mathcal{H}, D)$, in which $\mathcal{A}$ is the coordinate algebra and the operator D (playing the role of the Dirac operator) encodes the metric (cf. [16,17]). Thanks to the conformal covariant property of the spinor Dirac operator $\not{D}$, the conformal change of metric $g' = e^{-2h}g$, where $h \in C^\infty(M)$ real-valued, in the spectral setting can be achieved by replacing $\not{D}$ by $D_h = e^{h/2}\not{D}e^{h/2}$. The perturbed Dirac model $(C^\infty(M), L^2(\not{S}_g), D_h)$ via a conformal factor e^h is a natural example of twisted spectral triples of type III in [18].

Once the metric is fixed (as $\not{D}$ or D_h), local invariants, such as the scalar curvature function, are encoded in the heat kernel asymptotic:

$$\mathrm{Tr}(fe^{-t\not{D}^2}) \backsim_{t\searrow 0} \sum_{j=0}^{\infty} V_j(f, \not{D}^2)t^{(j-m)/2}, \quad \forall f \in C^\infty(M), \quad m = \dim M. \tag{1.1}$$

Each coefficient $V_j(f, \not{D}^2)$ is a spectral functional (in f). The associated local expressions $V_j(x, \not{D}^2) \in \Gamma(\mathrm{End}(\not{S}))$ (also called functional densities) are defined by the property:

$$V_j(f, \not{D}^2) = \varphi_0(fV_j(x, \not{D}^2)), \quad \forall f \in C^\infty(M),$$

here $\varphi_0(\psi) = \int_M \mathrm{Tr}_x(\psi)\mathrm{dvol}_g$, $\forall \psi \in \Gamma(\mathrm{End}(\not{S}))$, where Tr_x is the fiberwise trace. Up to an overall constant $(4\pi)^{-m/2}$, $V_0(x, \not{D}^2) = I$ which is related to the volume form and the next term recovers the scalar curvature function $\mathcal{S}_{\not{D}}$:

$$V_2(x, \not{D}^2) = -\frac{1}{12}\mathcal{S}_{\not{D}}. \tag{1.2}$$

See [19], Section 11.1 for a detailed discussion.

We are ready to state the main results of this paper. Assume that the dimension m of the manifolds is always even and greater than two, the requirement comes only from Lemma 6.1. Similar to the results on noncommutative two tori, the scalar curvature in the conformal geometry of Connes–Landi noncommutative manifolds is of the form:

$$\begin{aligned} R_{D_h} &= e^{(-m+2)h}\left(\mathcal{K}^{(m)}_{D_h}(\not\nabla)(\nabla^2 h) + \mathcal{H}^{(m)}_{D_h}(\not\nabla_{(1)}, \not\nabla_{(2)})(\nabla h\nabla h)\right)g^{-1} \\ &\quad + c_\Delta e^{(-m+2)h}\mathcal{S}_\Delta, \quad m = \dim M. \end{aligned} \tag{1.3}$$

Up to a volume factor e^{-mh}, (1.3) contains its Riemannian version (3.17) as the commutative limit. The noncommutative feature is reflected by the action of the modular derivation $\not\nabla = -2\mathrm{ad}_h$ through two local curvature functions $\mathcal{K}^{(m)}_{D_h}(u)$ and $\mathcal{H}^{(m)}_{D_h}(u, v)$, which are still begging for more conceptual understanding.

The dependence of the local curvature functions $\mathcal{K}_{D_h}(u)$ and $\mathcal{H}_{D_h}(u, v)$ on the dimension m is due to integration over the unit sphere $\mathbb{S}^{m-1}$ combined with some integration by parts arguments. It turns out that they are all contained in the germs (at $u = 0$) of function F and G defined in (3.6) and (3.7) respectively. In dimension four, the functions can be derived from F and G at $u = 0$; in dimension six, one needs to compute the first jet of F and G at $u = 0$; in dimension eight, one needs the second jet, etc.

For the Gaussian curvature of noncommutative two tori ([1], Theorem 4.8), there are two celebrated features for the local curvature functions: (1) The one variable function is the generating function of Bernoulli numbers. (2) There is an intriguing functional relation (Eq. 4.38 in [1] between them. The two facts both have generalizations in higher dimensions.

In dimension four, the one-variable local curvature function equals:

$$\mathcal{K}_{D_h}(u) = -\frac{1}{2}e^{u/2}\frac{\sinh(u/4)}{(u/4)}. \tag{1.4}$$

One can see the striking similarity to the analytic functions for the characteristic classes appeared in the Atiyah–Singer local index formula. The j-function $\sinh(x/2)/(x/2)$ defines the $\hat{A}$-class of a manifold while the exponential function gives rise to the Chern character of a vector bundle. It is not a coincident. Indeed, the j-function, or $f_1(u) = (e^u - 1)/u$, the reciprocal of the