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Symplectic scattering approach to gravitational lensing

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ABSTRACT

In this paper we apply the symplectic formalism to describe light deflection in general relativity as a scattering process. In the first part of the paper we present gravitational lensing in the action principle and in the Hamiltonian formulations. After defining the Souriau's symplectic scattering and introducing the mathematical properties in the infinitesimal case, we show that this formalism is an appropriate tool to study the deflection of a light signal in a gravitational field, not only in determining the scattering angle, but also the corresponding redshift and the Shapiro time delay. As examples we apply the symplectic scattering to lensing in weak fields and in the Swiss Cheese cosmological model. The conditions for the applicability of the infinitesimal scattering are derived. Finally we discuss its relation with the non infinitesimal case, we propose an extension of the results of the infinitesimal scattering and examine some properties of the non infinitesimal scattering in the hamiltonian formalism paying attention to the definition of photon sphere.

1. Introduction

In the last century after the exposition of the general theory of relativity, where all the gravitational phenomena can be ascribed to the curvature of spacetime, it was recognized that there is a close link between physics and geometry. Nevertheless a closer connection between physics and geometry was established previously by the development of symplectic geometry, after the pioneering work of Lagrange and Hamilton, which is the mathematics that underlies classical mechanics and optics. Nowadays symplectic geometry plays a crucial role in the formulation of many problems in classical and quantum physics [1], but also it has had a relevant role in mathematics by the so-called the "symplectization" of many mathematical branches as global analysis, mathematical physics, low-dimensional topology, dynamical systems, algebraic geometry, integrable systems and so on.

The formulation of symplectic geometry and its generalizations can be done in parallel with the Riemannian geometry in study of differentiable manifolds.

In symplectic geometry a standard symplectic form is introduced as the fundamental geometric object defining an antisymmetric product between two vectors, where in Riemann geometry scalar product between two vectors is generalized by introducing a metric. Unlike Riemannian manifolds, all symplectic manifolds of a given dimension are isomorphic in a neighborhood of each point. Due to the Darboux theorem there are no invariant structures like the curvature scalar.

One of the main predictions of general relativity is that a ray of light passing near a mass is deflected. This is a consequence of the fact that light travels along null geodesics in curved spacetime which differ from the corresponding null geodesics in

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a flat spacetime. This appears as a bending of light with respect to the initial trajectory. Light bending was considered the *experimentum crucis* of general relativity and motivated the expedition of 1919 to the Prince Islands, Africa, led by Arthur Eddington to record the position of the stars around the solar disk during the eclipse of the sun. The observation confirmed Einstein's predictions about the effect of light bending based on the theory of general relativity, published four years earlier, in 1915, and represented the first solid empirical evidence in favor of Einstein's theory. Nowadays this effect is commonly studied by observing the light of stars and of quasars passing close to any massive celestial body. Very recently [2] an Einstein Cross has been observed where light from a distant supernova passed through a galaxy forming four images.

Another significant effect related to the same class of phenomena is the gravitational time delay, or Shapiro delay [3], due to the fact that light signals take a longer time to move through a gravitational field than they would in the absence of that field. This effect has been confirmed by experiments [4,5] and with dramatical effects in the Einstein Cross observation, where the light of the supernova can be observed four times with time delays of decades.

Light bending is at the origin of the phenomena that go under the name of gravitational lensing giving rise to various phenomena such as Einstein rings, the formation of multiple images and the distortion of the same. For this reason gravitational lensing is an important observational tool to study the distributions of mass in the universe, as detecting the presence of dark mass in the galaxies, discovering extra solar system planets or even signaling the departures from the general relativity predictions implying either the presence of dark mass or alternatively the need of an extended gravitational theory. A full account of gravitational lensing, its properties can be found in the technical books [5,6] and in [7].

In this paper we show that light deflection can be described as the scattering a light ray produced by a massive object. More specifically by using the notion of symplectic scattering introduced by Souriau in [8] and subsequently completed by Guillemin and Sternberg in [9]. Scattering is defined as the mapping on the space of motions, which is a symplectic space of the free motions coordinatized by the position and the momentum of the unperturbed trajectories at a fixed time.

In particular weak lensing is described by the infinitesimal scattering which is the mapping obtained by the application of a tangent vector on the space of motions.

In other words light bending is described as the shifting of the motion of a light ray from a free motion to another one induced by the presence of a massive object.

By the Fermat's principle [5,10,11], one obtains lensing simply in terms of the trajectory on space. In our case we obtain a covariant description which encompasses the deflection angle together with the Shapiro time delay, as the shift of the source position, and in some particular cases the red shift .

The dual purpose of this paper is therefore to understand through an explicit physical example the mathematical concepts introduced with the (infinitesimal) symplectic scattering and to introduce a new algorithm for calculating the gravitational lensing.

The paper is organized in the following way. In Section 2 we present anew the calculations based on the action principle and in Section 3 we present briefly the Hamiltonian approach. Both these approaches are not covariant, thus we extend them conveniently to the symplectic approach. In Section 4 we present the general definition of symplectic scattering, this is constructed by introducing besides the fiber bundle (Y, M, π) over M, the fiber bundle (Y, U, ρ) with base manifold U such that the fibers are the free trajectories of the system (the integral curves of a two-form τ on Y)and where Y is a subset of the cotangent bundle $T^*\mathbf{R}^4$. Y was called by Souriau the evolution space while U was called the motion space. Scattering is therefore an application which relates elements of U with elements of U due to the presence of a scattering field. Direct computations can be made by applying these concepts to the infinitesimal symplectic scattering. This problem is studied in Section 5 and in our paper is related to the weak lensing. In this case we need to find a vector which associates to any element of U an infinitesimally close element on U. In Section 6 we obtain the explicit formula found in Sections 2 and 3 but they are completed and generalized by the expression of the scattering vectors. In Section 7 we generalize the results obtained to weak lensing where the gravitational field decays asymptotically. We see that if we take the distances of source and observer at infinity we recover the deflection angle formulas obtained in literature, see [5,6] and in [7] and an infinite time delay. But the expressions we obtain show also that if we consider just the finite trajectory between the source and the observer, we obtain also the time delay as a displacement of the source position and a redshift. In the second case we examine 7.3, we apply our method to the Swiss Cheese cosmological model, where the vacuole is an example of compact scattering region and the motions are straight lines considering that the Friedmann–Lemaitre–Robertson–Walker metrics are conformal to the Minkowski metric and the respective light cones coincide. We derive the formulas for the scattering angle, the time delay and the redshift.

In Section 8 we introduce some criteria to control the correctness of the results obtained previously. And finally in Section 9 we try to extend symplectic scattering to non infinitesimal cases. In 10 we discuss the results of our work and its perspectives.

2. The action principle and the Fermat principle

According to general relativity light propagates along null geodesics defined as the extremals of the action principle

$$\delta \left\{ \frac{1}{2} \int g_{ab} \dot{x}^a \dot{x}^b \right\} = 0, \tag{2.1}$$