Journal of Geometry and Physics 121 (2017) 279–287

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

On the Darboux integrability of the logarithmic galactic potentials

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A R T I C L E   I N F O

Article history:
Received 14 June 2013
Accepted 31 July 2017
Available online 7 August 2017

MSC:
primary 37J35
37K10

Keywords:
Logarithmic galactic potential
Polynomial integrability
Rational integrability
Darboux polynomials
Darboux first integrals
Invariant algebraic hypersurfaces

A B S T R A C T

We study the logarithmic Hamiltonians $H = (p_x^2 + p_y^2)/2 + \log(1 + x^2 + y^2/q^2)^{1/2}$, which appear in the study of the galactic dynamics. We characterize all the invariant algebraic hypersurfaces and all exponential factors of the Hamiltonian system with Hamiltonian H. We prove that this Hamiltonian system is completely integrable with Darboux first integrals if and only if $q = \pm 1$.

1. Introduction and statement of the main results

The potential

$$V = \frac{1}{2}\log\left(R^2 + x^2 + \frac{y^2}{q^2}\right),$$

where $q \in \mathbb{R} \setminus \{0\}$ is called the *logarithmic potential*. It has an absolute minimum and reflection symmetry with respect to both axes. This potential is relevant in problems of galactic dynamics as a model for elliptical galaxies. More precisely, it is a model of a core embedded in a dark matter halo, with R being the core radius. Without loss of generality we can assume that $R = 1$, and the energy can take any non-negative value. The parameter q is the ellipticity of the potential, which ranges in the interval $0.6 \le q \le 1$. Lower values of q have no physical meaning and greater values of q are equivalent to reverse the role of the coordinate axes. In this paper, to make a complete and deep study of the Darboux integrability of such a potential we will consider that $q \in \mathbb{R} \setminus \{0\}$. This model has been intensively investigated from different dynamical and physical points of view by several authors, see for instance [1–5].

We consider the logarithmic Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\log\left(1 + x^2 + \frac{y^2}{q^2}\right), \quad q \in \mathbb{R} \setminus \{0\},$$

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http://dx.doi.org/10.1016/j.geomphys.2017.07.019

its Hamiltonian system is

$$\begin{aligned}\dot{x} &= -p_x,\\ \dot{y} &= -p_y,\\ \dot{p}_x &= \frac{x}{1+x^2+y^2/q^2},\\ \dot{p}_y &= \frac{y}{q^2(1+x^2+y^2/q^2)},\end{aligned} \tag{1}$$

where the dot indicates derivative with respect to time t. Note that this Hamiltonian system (1) has two degrees of freedom.

The main aim of this paper is to study the existence of first integrals of system (1). The vector field X associated to system (1) is

$$X = -p_x\frac{\partial}{\partial x} - \frac{p_y}{q}\frac{\partial}{\partial y} + \frac{x}{1+x^2+y^2/q^2}\frac{\partial}{\partial p_x} + \frac{y}{q^2(1+x^2+y^2/q^2)}\frac{\partial}{\partial p_y}.$$

Let $U \subset \mathbb{R}^2$ be an open set. We say that the non-constant function $F : \mathbb{R}^2 \to \mathbb{R}$ is a *first integral* of a vector field X on U, if $F(x(t), y(t), p_x(t), p_y(t)) =$ constant for all values of t for which the solution $(x(t), y(t), p_x(t), p_y(t))$ of X is defined on U. Clearly F is a first integral of X on U if and only if $XF = 0$ on U.

We say that the functions $F_1, \dots, F_n$ are in *involution* if $\{F_i, F_j\} = 0$ for all $i \neq j$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket. Moreover, they are *independent* if the one-forms $dF_1, \dots, dF_n$ are linearly independent over a full Lebesgue measure subset of the common definition domain of F_j for $j = 1, \dots, n$. By definition, a Hamiltonian system with n degrees of freedom having n independent first integrals in involution is *completely integrable*, see for more details [6].

Note that system (1) is completely integrable, if and only if there exists a first integral linearly independent and in involution with H. We have the following result, whose proof follows by direct computations.

Proposition 1. *When $q = \pm 1$ the Hamiltonian system (1) is completely integrable with the first integrals H and $H_1 = yp_x - xp_y$.*

From now on we will restrict to the case $q \neq \pm 1$. Doing the change of time $dt = (1+x^2+y^2/q^2)\,ds$, system (1) becomes

$$\begin{aligned}x' &= -p_x(1+x^2+y^2/q^2),\\ y' &= -p_y(1+x^2+y^2/q^2),\\ p_x' &= x,\\ p_y' &= \frac{y}{q^2},\end{aligned}$$

where the prime denotes derivative with respect to the new time s.

Taking the notation $Y = y/q$, $Q = 1/q^2 > 0$, $P_Y = qp_y$ we get

$$\begin{aligned}x' &= -p_x(1+x^2+Y^2),\\ Y' &= -QP_Y(1+x^2+Y^2),\\ p_x' &= x,\\ P_Y' &= Y.\end{aligned}$$

We write the previous system again as

$$\begin{aligned}x' &= -p_x(1+x^2+y^2),\\ y' &= -Qp_y(1+x^2+y^2),\\ p_x' &= x,\\ p_y' &= y.\end{aligned} \tag{2}$$

The vector field associated to system (2) is

$$X = -p_x(1+x^2+y^2)\frac{\partial}{\partial x} - Qp_y(1+x^2+y^2)\frac{\partial}{\partial y} + x\frac{\partial}{\partial p_x} + y\frac{\partial}{\partial p_y}.$$

Note that system (2) has the first integral

$$H_0 = (1+x^2+y^2)e^{p_x^2+Qp_y^2}. \tag{3}$$

From now on $Q \neq 1$.

The aim of this paper is to study the existence of additional first integrals of system (2) which are linearly independent with H_0 and that can be described by functions of Darboux type (see (7)). Note that one of the main tools for studying the dynamics of the differential system (2) is to know the existence of an additional independent first integral for some values