Journal of Geometry and Physics 121 (2017) 288–296

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

# Log-canonical coordinates for Poisson brackets and rational changes of coordinates

John Machacek, Nicholas Ovenhouse

*Department of Mathematics, Michigan State University, USA*

ARTICLE INFO

*Article history:*
Received 4 April 2017
Received in revised form 26 July 2017
Accepted 31 July 2017
Available online 7 August 2017

*MSC:*
primary 17B63
secondary 13F60
53D17

*Keywords:*
Poisson algebras
Poisson varieties
Log-canonical coordinates
Cluster algebras

ABSTRACT

Goodearl and Launois have shown in Goodearl and Launois (2011) that for a log-canonical Poisson bracket on affine space there is no rational change of coordinates for which the Poisson bracket is constant. Our main result is a proof of a conjecture of Michael Shapiro which states that if affine space is given a log-canonical Poisson bracket, then there does not exist any rational change of coordinates for which the Poisson bracket is linear. Hence, log-canonical coordinates can be thought of as the simplest possible algebraic coordinates for affine space with a log-canonical coordinate system. In proving this conjecture we find certain invariants of log-canonical Poisson brackets on affine space which linear Poisson brackets do not have.



## 1. Introduction

Cluster algebras were originally defined by Fomin and Zelevinsky [1] to study total positivity and dual canonical bases in semisimple groups. Since their inception, connections between cluster algebras and many areas of algebra and geometry have been found. One such connection is with Poisson geometry. Gekhtman, Shapiro, and Vainshtein [2] have studied Poisson structures compatible with cluster algebras. In this compatibility, the cluster variables give log-canonical coordinates for the Poisson bracket, while the mutations give birational transformations preserving the log-canonicity. We will study log-canonical Poisson brackets under rational changes of coordinates. Our main result is Theorem 12 where we show that log-canonical coordinates are analogous to Darboux coordinates for rational algebraic functions in the sense that the Poisson bracket takes the simplest form in these coordinates.

### 1.1. Poisson algebras and Poisson geometry

Let $P$ be an associative algebra. A *Poisson bracket* on $P$ is a skew-symmetric bilinear map $\{\cdot,\cdot\} : P \times P \to P$ such that for any $a, b, c \in P$ both the *Leibnitz identity*

$$\{ab, c\} = a\{b, c\} + \{a, c\}b$$

and the *Jacobi identity*

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$$

hold. A *Poisson algebra* is pair $(P, \{\cdot,\cdot\})$ where $P$ is an associative algebra and $\{\cdot,\cdot\}$ is a Poisson bracket.

*E-mail addresses:* machace5@math.msu.edu (J. Machacek), ovenhou3@math.msu.edu (N. Ovenhouse).

http://dx.doi.org/10.1016/j.geomphys.2017.07.023
0393-0440/

Notice that $\{\cdot,\cdot\}$ makes $P$ a Lie algebra. So, we get the adjoint representation of $P$ on itself sending $a \in P$ to $\mathrm{ad}_a \in \mathrm{End}(P)$, where $\mathrm{ad}_a(b) = \{a, b\}$. Note that the Jacobi identity implies that $\mathrm{ad}_a$ is a Lie algebra derivation. Also observe that $\mathrm{ad}_a$ is a derivation of the associative algebra $P$ by the Leibniz identity. If $a \in P^*$ is a unit, then the Leibniz identity implies that $\mathrm{ad}_{a^{-1}} = -a^{-2}\mathrm{ad}_a$. In particular, this implies that if $\{a, b\} = 0$ for some $a \in P^*$ and $b \in P$, then $\{a^{-1}, b\} = 0$.

Let $M$ be a smooth manifold, and let $C^\infty(M)$ denote its algebra of smooth functions. A *Poisson structure* on $M$ is a bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that $(C^\infty(M), \{\cdot,\cdot\})$ is a Poisson algebra. In this case we call $(M, \{\cdot,\cdot\})$ a *Poisson manifold*. For local coordinates $(x_1,\dots,x_n)$ and $f, g \in C^\infty(M)$ the Poisson bracket is given by

$$\{f, g\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\{x_i, x_j\} \tag{1}$$

and so the bracket is completely determined by the $\binom{n}{2}$ "structure functions" $\{x_i, x_j\}$, for $i < j$. Following [2], a system of coordinates $(x_1,\dots,x_n)$ is called *log-canonical* with respect to a Poisson bracket $\{\cdot,\cdot\}$ if there is a matrix of scalars $\Omega = (\omega_{ij})$ (necessarily skew-symmetric) such that the structure functions are given by $\{x_i, x_j\} = \omega_{ij}x_ix_j$. We note here that this Poisson structure goes by many names in the literature. For example, it is called a "*diagonal Poisson structure*" in [3], "*Poisson n-space*" in [4], and a "*semi-classical limit of quantum affine space*" in [5].

In general, the local structure of Poisson manifolds is described by the following theorem of Weinstein.

**Theorem** ([6]). *Let $M$ be a Poisson manifold, and $p \in M$. Then there exists a neighborhood $U$ containing $p$ with coordinates $(x_1, y_1,\dots,x_r, y_r, z_1,\dots,z_s)$, such that the bracket takes the form*

$$\{x_i, x_j\} = \{y_i, y_j\} = \{x_i, z_j\} = \{y_i, z_j\} = 0$$

$$\{x_i, y_j\} = \delta_{ij}$$

$$\{z_i, z_j\} = \varphi_{ij}$$

*where $\varphi_{ij} \in C^\infty(U)$ depend only on $z_1,\dots,z_s$, and $\varphi_{ij}(p) = 0$.*

**Example.** If $(M^{2n}, \omega)$ is a symplectic manifold, then there is a standard Poisson structure induced by $\omega$. In this special case, Weinstein's theorem is the classical Darboux theorem which says that locally $\omega$ has the form

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

The local coordinates $(x_1, y_1,\dots,x_n, y_n)$ are commonly called "canonical coordinates" or "Darboux coordinates".

Note that on a smooth Poisson manifold with a log-canonical system of coordinates $(x_1,\dots,x_n)$ the system of coordinates $(y_1,\dots,y_n) = (\log x_1,\dots,\log x_n)$, defined on the open set where all $x_i$ are positive, is similar to a system of canonical coordinates in the sense that the structure functions

$$\{y_i, y_j\} = \{\log x_i, \log x_j\} = \omega_{ij}$$

are all constants. This is indeed the intuition behind the terminology "log-canonical". From Theorem 7 it will follow that there does not exist any rational change of coordinates on any Zariski open subset such that the structure functions are constant in the new coordinates.

Similarly, let $M$ be an algebraic variety and $\mathcal{O}(M)$ its algebra of regular functions. If there is a bracket making $\mathcal{O}(M)$ into a Poisson algebra, then we call $(M, \{\cdot,\cdot\})$ a *Poisson variety*. Suppose there is a system of coordinates $(x_1,\dots,x_n)$ on some Zariski open subset of a Poisson variety $M$, then the bracket is given by Eq. (1) just as in the smooth case (see for example [3] for details). We wish to investigate whether such a "simplification" of the structure functions is possible (analogous to the simplification in the Darboux/Weinstein Theorem, in the sense that all structure functions become lower degree polynomials), allowing only birational change-of-coordinates. It is suggested/conjectured in [7] that there are not canonical coordinates in general for an arbitrary Poisson variety, but no specific counterexample has been demonstrated. In [8], it was shown that affine space with a log-canonical bracket is such a counterexample. We wish to demonstrate that this same example has the additional property that no rational change of coordinates can make the structure functions linear. The following example is given in [7] and demonstrates some of the nuances of the problem of finding canonical coordinates on an open set of a Poisson variety.

**Example** ([7]). Consider affine space $\mathbb{C}^2$ with coordinates $(x, y)$ and Poisson bracket given by $\{x, y\} = x$. Viewing $\mathbb{C}^2$ as a smooth manifold, there is a system of canonical local coordinates $(\log x, y)$ that is *not* algebraic. However, there is also $\left(\frac{1}{x}, -xy\right)$ which is a system of canonical local coordinates that *is* algebraic. That is, a system of canonical coordinates consisting of rational functions in $x$ and $y$ defined on the Zariski-open subset $\{(x, y) : x \neq 0\}$ of the variety $\mathbb{C}^2$. The example illustrates that there do exist Poisson varieties which admit a rational coordinate change on an open subset which makes the structure functions constant.