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Differential invariants for flows of viscid fluids

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ABSTRACT

Algebras of symmetries and the corresponding algebras of differential invariants for 3D-flows of viscid newtonian fluids are given. Their dependence on thermodynamical states of media is studied.

1. Introduction

In this paper we study differential invariants of 3D-flows of compressible viscid newtonian fluids with respect to their symmetry group.

The system of differential equations governing such flows consists of the following equations:

$$\begin{cases} \rho \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\operatorname{grad} p + \eta \Delta \mathbf{u} + \left(\zeta + \dfrac{\eta}{3}\right) \operatorname{grad}(\operatorname{div} \mathbf{u}) + \mathbf{g}\rho, \\ \dfrac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \operatorname{div} \mathbf{u} = 0, \\ T\rho \dfrac{\mathrm{D}s}{\mathrm{D}t} = k\Delta T + \displaystyle\sum_{i,j} \sigma_{ij} \dfrac{\partial u_i}{\partial x_j}, \end{cases} \tag{1}$$

where the vector $\mathbf{u} = (u_1, u_2, u_3)$ is the flow velocity, p, ρ, s, T are the pressure, density, entropy, temperature of the fluid respectively, and $\mathbf{g} = (0, 0, g)$ is the gravity force.

The derivative

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \operatorname{grad} f$$

is a material or substantial derivative and $\Delta = \operatorname{div}\operatorname{grad}$ is the Laplace operator.

Here σ_{ij} is a viscous stress tensor (see, for example, [1]) defined in the following way:

$$\sigma_{ij} = \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \frac{\partial u_k}{\partial x_k} \right) + \zeta \delta_{ij} \frac{\partial u_k}{\partial x_k}.$$

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The thermal conductivity k and the viscosities ζ and η are assumed to be constants.

The first equation of system (1) is the 3-dimensional Navier–Stokes equation, the second is the expression of the conservation mass law and the third is the general equation of heat transfer. See also [1] for details.

Further we use coordinates (x, y, z) instead of (x_1, x_2, x_3) and (u, v, w) for flow velocity (u_1, u_2, u_3).

Note that system (1) is incomplete. In order to complete it we add two additional equations using the thermodynamics of the medium. In the paper [2] we give the classification of thermodynamic states and the corresponding Lie algebras for the cases when the thermodynamic state admits a one- or two-dimensional symmetry algebra. This classification is also valid in our case.

The paper is organized as follows. In Section 2 we recall main thermodynamic notions in a geometrical form more convenient for our purposes. Thus, the thermodynamic states will be given by Lagrangian surfaces or by two differential equations with an additional compatibility condition.

In Section 3 we discuss symmetry Lie algebras of the Navier–Stokes system and especially their dependence on the thermodynamic state. In general, the symmetry algebra consists of pure geometrical and thermodynamic parts. The geometrical part represents the symmetry (4) with respect to a group of motions, Galilean transformations and time shifts. The thermodynamic part strongly depends on the symmetries of the thermodynamic state.

The smallest symmetry algebra (of dimension 10) is realized for general thermodynamic states. This algebra depends on the geometry of the medium, and we call the corresponding differential invariants kinematic.

In Section 4 we give a complete description of the algebra of kinematic invariants. Depending on a symmetry of the thermodynamic states we get a bigger symmetry algebra and therefore a smaller algebra of invariants (we call them Navier–Stokes invariants). We give also a description of these algebras for the case of the thermodynamic states with one- or two-dimensional symmetry algebras. It follows that the Navier–Stokes invariants are obtained by adding some constraints on the kinematic ones. Also observe that the Navier–Stokes invariants give us the complete information about the flow as well as the medium.

Many of the computations in this paper were done in Maple with the remarkable Differential Geometry package [3] by I. Anderson and his team. Maple files with the most important computations in this paper can be found on the web-site http://d-omega.org.

2. Thermodynamics

As we have seen, the PDE system (1) is not complete. It has 5 equations for 7 unknown functions. To complete it we need two additional relations on the thermodynamic quantities used in the system.

Namely, in our case we have the following thermodynamic quantities: the specific volume ρ^{-1}, the specific entropy s and the specific internal energy ϵ.

Geometrically, the main thermodynamical relations can be formulated as follows.

Let us consider a 5-dimensional contact manifold $\Phi = \mathbb{R}^5$ equipped with coordinates $(p, \rho, s, T, \epsilon)$ and the contact 1-form

$$\theta = d\epsilon - Tds - \frac{p}{\rho^2}d\rho.$$

Then the thermodynamical states are a 2-dimensional Legendrian manifold L, i.e. such surface $L \subset \Phi$, that the first law of thermodynamics $\theta\Big|_L = 0$ holds.

We will consider this case, when the functions (ρ, s) are coordinates on the manifold L. Then this surface can be defined by the structure equations:

$$\epsilon = \epsilon(\rho, s), \quad T = \frac{\partial \epsilon}{\partial s}, \quad p = \rho^2\frac{\partial \epsilon}{\partial \rho}.$$

Remark that the Navier–Stokes system (1) does not depend on the specific energy ϵ.

In order to eliminate the internal energy ϵ from the description of the thermodynamic states we consider the projection $\phi : \mathbb{R}^5 \to \mathbb{R}^4, \phi : (p, \rho, s, T, \epsilon) \longmapsto (p, \rho, s, T)$. The restriction of the map ϕ on the state surface L is a diffeomorphism on the image $\bar{L} = \phi(L)$ and the surface $\bar{L} \subset \mathbb{R}^4$ is a Lagrangian manifold in the 4-dimensional symplectic space $\mathbb{R}^4$ equipped with the structure form

$$\Omega = ds \wedge dT + \rho^{-2}d\rho \wedge dp.$$

Moreover, the specific energy ϵ of the state can be reconstructed (up to a constant) from the Lagrangian surface $\bar{L}$.

Therefore, equivalently, the thermodynamic states could be considered as the Lagrangian submanifolds in the symplectic space $(\mathbb{R}^4, \Omega)$.

Thus, if we define the two-dimensional surface $\bar{L}$ by the equations

$$\begin{cases} F(p, \rho, s, T) = 0, \\ G(p, \rho, s, T) = 0, \end{cases} \tag{2}$$

then the condition for the surface $\bar{L}$ to be Lagrangian has the following form:

$$[F, G] = 0 \text{ on } \bar{L}, \tag{3}$$