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Review

Minimal surfaces in Lorentzian Heisenberg group and Damek–Ricci spaces via the Weierstrass representation

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ABSTRACT

In this paper we discuss a Weierstrass type representation for minimal surfaces in the 3-dimensional Heisenberg group and in the 4-dimensional Damek–Ricci spaces, endowed with left invariant Riemannian or Lorentzian metrics. For the case of spacelike surfaces we employ the complex analysis, and for timelike surfaces our approach makes use of the paracomplex analysis. Then, we exhibit various examples of spacelike and timelike minimal surfaces in these spaces.

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1. Introduction

The Weierstrass representation formula is an important tool in the study of minimal surfaces in $\mathbb{R}^n$ since, on one hand it is a machinery to produce examples, and on the other hand it allows to bring into the theory the powerful tools of the complex analysis.

The local version of the Weierstrass representation formula has been extended for minimal surfaces in a Riemannian manifold in [1] and in Lorentzian manifolds in [2]. However, the differential equations involved are quite difficult to handle and, in general, we need an ad hoc argument for each ambient space.

The aim of this paper is to describe these arguments in the case of the 3-dimensional Heisenberg group and in the 4-dimensional Damek–Ricci group, equipped with a left invariant Riemannian or Lorentzian metric, and to produce new interesting examples of minimal surfaces in these spaces.

The Heisenberg groups $\mathbb{H}_{2n+1}$ are well known in theoretical physic and the Damek–Ricci spaces are semidirect products of Heisenberg groups with the real line. The latter were introduced by Damek and Ricci in [3] (see also [4]) to give a negative answer to a question posed by Lichnerowicz [5]: "is a harmonic Riemannian manifold necessarily a symmetric space?" In [3] it is shown that they are globally harmonic manifolds in infinitely many dimensions greater than or equal to 8 which are not locally symmetric.

Besides a left invariant Riemannian metric, the Damek–Ricci spaces may be equipped with left invariant Lorentzian metrics in essentially two ways: putting a Riemannian metric on the Heisenberg factor and a negative definite metric on the $\mathbb{R}$ factor, or a Lorentzian metric in the Heisenberg factor and a positive definite metric on $\mathbb{R}$.

Our paper is also motivated by the fact that there are not many examples of application of the generalized Weierstrass representation formula in codimension bigger than one. A typical employment of this result is the existence and uniqueness of the solution of the Björling problem in $\mathbb{R}^3$, that was proposed by Björling in 1844 (see [6]) and solved by Schwarz in 1890 (see [7]). Some extensions of this problem in others three-dimensional ambient spaces have been proposed and solved in [8,9], [10,17]. In codimension bigger than one, there are [11–13] and, also, [14] where the authors use the Weierstrass representation to construct minimal surfaces in the 4-dimensional Damek–Ricci spaces equipped with a left invariant Riemannian metric. However their calculations contain some errors.

This paper is organized as follows: in Section 2 we recall the basic facts about the generalized Weierstrass representation formula for minimal surfaces in Riemannian and Lorentzian manifolds. For the timelike surfaces in a Lorentzian manifold is employed the paracomplex analysis. In Section 3 we will use this setting to construct examples of minimal surfaces in the Lorentzian 3-dimensional Heisenberg group $\mathbb{H}_3$. Also, we express the first fundamental form, the Gauss map and the Gauss curvature of a minimal surface in $\mathbb{H}_3$ in terms of its Weierstrass data. In Section 4 we will describe the geometry of the Damek–Ricci spaces, considering in the Lorentzian case those of the first and the second kind. Then, in the next sections we discuss the generalized Weierstrass representation for minimal surfaces in the 4-dimensional Damek–Ricci spaces endowed with a left invariant Riemannian or Lorentzian metric and, also, we produce new examples of minimal surfaces in these spaces.

2. The Weierstrass representation formula

In this section, we will briefly discuss the generalized Weierstrass representation formula starting with the Riemannian case. Since the considerations will be local, we can suppose that the ambient manifold is $\mathbb{R}^n$ with a Riemannian metric $g = (g_{ij})$. The following result is taken from [1, Theorem 2.1].

Theorem 2.1 ([1]). *Let $\Omega \subseteq \mathbb{C}$ be an open set and let $f : \Omega \to \mathbb{R}^n$ be a conformal minimal immersion. Let $\{u, v\}$ be conformal coordinates in Ω and $z = u + i\,v$ the complex parameter. Consider the complex tangent vector*

$$\frac{\partial f}{\partial z} := \frac{1}{2}\Big(\frac{\partial f}{\partial u} + i\,\frac{\partial f}{\partial v}\Big) = \sum_{i=1}^{n} \phi_i\,\frac{\partial}{\partial x_i},$$

then its Euclidean components ϕ_i, $i = 1, \ldots, n$, satisfy the following conditions:

1. $\sum_{i,j=1}^{n} g_{ij}\,\phi_i\,\phi_j \neq 0$,
2. $\sum_{i,j=1}^{n} g_{ij}\,\phi_i\overline{\phi_j} = 0$,
3. $\frac{\partial \phi_i}{\partial \bar z} + \sum_{j,l=1}^{n} \Gamma^i_{jl}\,\phi_j\,\overline{\phi_l} = 0$,

where Γ^i_{jl} are the Christoffel symbols of g. Moreover, if Ω is simply connected, the functions

$$f_i := 2\,\mathcal{R}e \int \phi_i\,\mathrm{d}z, \qquad i = 1, \ldots, n, \tag{1}$$