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Multiplicative quiver varieties and generalised Ruijsenaars–Schneider models

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ABSTRACT

We study some classical integrable systems naturally associated with multiplicative quiver varieties for the (extended) cyclic quiver with m vertices. The phase space of our integrable systems is obtained by quasi-Hamiltonian reduction from the space of representations of the quiver. Three families of Poisson-commuting functions are constructed and written explicitly in suitable Darboux coordinates. The case $m = 1$ corresponds to the tadpole quiver and the Ruijsenaars–Schneider system and its variants, while for $m > 1$ we obtain new integrable systems that generalise the Ruijsenaars–Schneider system. These systems and their quantum versions also appeared recently in the context of supersymmetric gauge theory and cyclotomic DAHAs (Braverman et al. [32,34,35] and Kodera and Nakajima [36]), as well as in the context of the Macdonald theory (Chalykh and Etingof, 2013).

1. Introduction

Among the most powerful geometric techniques in the theory of integrable systems is the method of Hamiltonian (or symplectic) reduction. Invented initially for reducing the degrees of freedom in Hamiltonian systems with symmetries, the concept of a moment map and symplectic reduction have since found a multitude of uses beyond their initial scope. One of the earlier examples of a Hamiltonian reduction was given by Kazhdan, Kostant and Sternberg [1], who demonstrated how to obtain the celebrated Calogero–Moser system from a very simple system on $T^*\mathfrak{gl}_n$. Since then, many integrable systems have been obtained or interpreted by similar methods. Among those is a remarkable generalisation of the Calogero–Moser system introduced by Ruijsenaars and Schneider [2]. The latter system was interpreted in terms of an infinite-dimensional symplectic reduction by Nekrasov [3], extending his earlier work with A. Gorsky [4]. Hyperbolic Ruijsenaars–Schneider system can also be obtained by a finite-dimensional reduction in the spirit of [1], as was demonstrated by Fock and Rosly [5]. Although Fock and Rosly employ Hamiltonian (or Poisson) reduction, their construction allows an interpretation in terms of *quasi-Hamiltonian* reduction. We recall that the method of quasi-Hamiltonian reduction was developed by Alekseev, Malkin and Meinrenken in [6], see also [7]. The main difference is that the reduction is performed on a space which may not be symplectic, and the moment map takes values in the Lie group rather than the Lie algebra. Not attempting at a comprehensive review, we refer the reader to some of the more recent papers [8–13], where the Ruijsenaars–Schneider model and its variants are treated by the method of (quasi-)Hamiltonian reduction, and where further references can be

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found. Let us also mention an alternative geometric approach to many-body problems by Krichever [14], in which the Lax matrix structure plays the central role instead, and the Hamiltonian picture is derived from that, cf. [15–17].

From yet another perspective, a unified view onto the Calogero–Moser and Ruijsenaars–Schneider system can be achieved by noticing that in both cases the reduction is done on (the cotangent bundle to) the space of representations of a one-loop quiver. Such a view onto the (complexified) Calogero–Moser system was brought forward by G. Wilson's work [18] relating the rational Calogero–Moser system, adelic Grassmannian, and the KP hierarchy, and it has been further deepened in [19,20], cf. [21,22]. The present paper stems from a natural idea to look for a generalisation of Wilson's results for more complicated quivers. We recall that with any quiver Nakajima associates in [23] a class of symplectic quotients called *quiver varieties*. There exists also a multiplicative version of quiver varieties, introduced by Crawley-Boevey and Shaw [24] and interpreted via quasi-Hamiltonian reduction by Van den Bergh [25]. Affine Dynkin quivers are a particularly well studied class, and a large part of Wilson's (and Berest–Wilson's) results have already been extended to this case by Ginzburg, Baranovsky and Kuznetsov [26,27] (see also [28,29]). However, the multiplicative case has not been systematically looked at, apart from the already mentioned case of a one-loop quiver. This was the main motivation behind our work. We will focus on the link to integrable particle dynamics; other aspects of the Calogero–Moser correspondence will be discussed elsewhere. Our main result is a construction of new generalisations of the Ruijsenaars–Schneider system, related to cyclic quivers. This is achieved by performing a quasi-Hamiltonian reduction on the space of representations of the associated multiplicative preprojective algebras Λ^q of Crawley-Boevey and Shaw [24]. Our main tool is the formalism of double (quasi-)Poisson algebras due to Van den Bergh [25,30]. The constructed integrable systems come equipped with a complete phase space (represented by a suitable multiplicative quiver variety), and the associated Hamiltonian dynamics can be explicitly integrated. By constructing Darboux coordinates on the phase space, we express the new integrable Hamiltonians in coordinates, which then allows us to identify them as generalisations of the Ruijsenaars–Schneider system. For non-multiplicative quiver varieties, analogous integrable systems can be identified with the rational Calogero–Moser system for $W = S_n \wr \mathbb{Z}_m$, see [29]. Thus, the systems constructed in the present work can be considered as q-analogues of the rational Calogero–Moser system for such W. The very fact that such q-analogues exist is somewhat surprising. Indeed, the group $S_n \wr \mathbb{Z}_m$ is noncrystallographic for general m, while trigonometric or hyperbolic Calogero–Moser systems and their q-analogues are usually expected to have a crystallographic symmetry group.

Interestingly, quantum versions of some of these systems appeared in a seemingly unrelated context in [31], where they were called *twisted Macdonald–Ruijsenaars systems*. By computing some of these quantum Hamiltonians explicitly, we are able to see this relationship in the case of the quiver with two vertices, although a direct comparison in the general case is more difficult. However, a recent work by Braverman, Etingof and Finkelberg [32], which appeared while we were finishing the present paper, clarifies this connection rather remarkably. It introduces a cyclotomic version of the double affine Hecke algebra (DAHA) in type A. Inside the cyclotomic DAHA there are three natural commutative subalgebras and they give rise to quantum integrable systems, in the same way as the usual DAHA can be used to produce the Macdonald–Ruijsenaars operators. The classical versions of these systems correspond to the $q = 1$ limit of the cyclotomic DAHA (cf. [33] for the case of the usual DAHA), and this leads to the multiplicative quiver varieties for the cyclic quiver. Thus, the integrable systems constructed in [32] coincide (on the classical level) with those constructed by us. The interpretation of these integrable systems via the cyclotomic DAHA in [32] allows to explain their relationship to the twisted Macdonald–Ruijsenaars systems from [31] in type A. Our methods are quite different in comparison, and they allow us to find explicit formulas for the corresponding classical Hamiltonians and integrate the Hamiltonian flows (the approach via the cyclotomic DAHA in [32] is less explicit). Curiously, these Hamiltonians become much simpler under the Cherednik–Fourier transform. In this form they appeared in the work of Braverman, Finkelberg, and Nakajima [34,35] on the quantised Coulomb branch of quiver gauge theories, see also a related work of Kodera and Nakajima [36]. This can also be seen from our formulas at the classical level, when the Cherednik–Fourier transform becomes the angle–action transform studied by Ruijsenaars [37]. See Section 5 for more details. Apart from being more explicit compared to [32], our approach also has an advantage of being better suited for studying spin versions of the Ruijsenaars–Schneider system and its generalisations; this will be a subject of a future work.

The structure of the paper is as follows. In Section 2 we first describe the general formalism of double Poisson brackets and quasi-Poisson algebras due to Van den Bergh [25], and then exemplify it for the multiplicative quiver varieties. Section 3 looks at the tadpole quiver, explaining how to obtain the hyperbolic Ruijsenaars–Schneider system by quasi-Hamiltonian reduction. In Section 4 we consider the multiplicative quiver varieties (Calogero–Moser spaces) for the framed cyclic quiver with m vertices. We introduce three Poisson commuting families of functions on those Calogero–Moser spaces, and integrate the corresponding Hamiltonian flows. We then write these Hamiltonians in suitable Darboux coordinates, identifying them as generalisations of the hyperbolic Ruijsenaars–Schneider system. Finally, in Section 5 we discuss the relationship between our work and the results of [31] and [32,34–36]. In particular, we were able to write explicitly the integrable quantum Hamiltonians from [32] in the case of a quiver with two vertices. The paper finishes with three appendices containing some of the more technical proofs.

2. Preliminaries

In this section we first recap the theory of double Poisson brackets and double (quasi-)Poisson algebras due to Van den Bergh [25]. We then describe a concrete example of this formalism, related to multiplicative preprojective algebras and multiplicative quiver varieties of Crawley-Boevey and Shaw [24]. We will follow the notation of the papers [25,30], where the reader can find many more details.