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Long-time existence of a geometric flow on closed Hessian manifolds

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ABSTRACT

In this paper we study Hessian manifolds. We define a flow, which we call the Hessian flow, to study the existence of Einstein–Hessian metrics on Hessian manifolds. The flow will considered as a real Monge–Ampére equation and we prove the short-time existence, the global existence and the uniqueness of it.

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1. Introduction

A Riemannian metric on a flat affine manifold is called a Hessian metric if it is locally expressed by the Hessian of a smooth function. A triple of (M, D, g) with an affine manifold M, flat connection D, and Hessian metric g is called a Hessian manifold. Hessian structure was first studied by Koszul [1], then Vinberg [2] considered this metric on convex cone and S.Y. Cheng and S.T. Yau [3] used Hessian metrics to solve Monge–Ampére equations. Hessian geometry finds connection with various fields of sciences in both Pure mathematical fields such as affine differential geometry, homogeneous spaces, cohomology and applied sciences such as physics, statistics and information geometry. It is well-known that many important smooth families of probability distributions (e.g. exponential families) admit Hessian structures [4]. The geometry of Hessian manifolds is also deeply related to Kählerian manifolds. A Hessian structure is formally analogous to a Kählerian structure because a Kählerian metric is locally expressed as a complex Hessian of a function with respect to the holomorphic coordinate systems. The relation of these two kinds of manifolds was motivation of defining a flow on Hessian manifolds similar to Kähler–Ricci flow.

One of the most important questions in Riemannian geometry is that whether a manifold admits a canonical metric? Geometric flows, as a class of important geometric partial differential equations, are used to answer this question in some cases. In 1982 Richard Hamilton [5] introduced the so called Ricci flow, for solving Thurston's conjecture and classified compact 3-manifolds of positive Ricci curvature as spherical space forms. Later in 1985 Cao [6] studied the Ricci flow on complex Kählerian manifolds and showed that on closed Kähler manifolds this flow exists for all time and converges to

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an Einstein–Kähler metric under normalized Kähler–Ricci flow. Shi [7] generalized this result to complete noncompact Kählerian manifolds.

In Hessian manifolds we intend to deform Hessian structure and alter Hessian metric on M to a canonical metric with good behavior, for this reason we define a PDE, as a geometric flow that evolves the metric tensor on manifold, and we will show the global existence of the flow.

2. Preliminaries

This section includes some definitions and facts that we need in following sections.

2.1. Geometry of Hessian manifolds

Definition 2.1.1. Let (M^m, D) be a flat affine manifold, if a Riemannian metric g on M has a local expression $g = Ddu$ for $u \in C^\infty(M)$, g is called a Hessian metric, (D, g) a Hessian structure and (M, D, g) a Hessian manifold and u is said to be a potential function of (D, g), that is,

$$g_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j},$$

where $(x^1, x^2, \dots, x^m)$ is an affine coordinate system with respect to D (i.e. $Ddx^i = 0$).

At the following, in our notation we use this coordinate system.

Proposition 2.1.1 ([8]). *Let (M, D) be a flat manifold, then (M, D, g) is a Hessian manifold if and only if*

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{kj}}{\partial x^i}, \qquad \forall i, j, k = 1, \dots, m.$$

Let (M, D) be a flat manifold, g a Riemannian metric, and ∇ the Levi-Civita connection of g. We denote the difference tensor of D and ∇ by

$$\gamma := \nabla - D.$$

It should be remarked that the components of γ with respect to affine coordinate systems of D coincide with the Christoffel symbols of ∇.

Definition 2.1.2 ([8]). Let (D, g) be a Hessian structure. A tensor field Q of type(1,3) defined by covariant differential of γ with respect to D

$$Q = D\gamma,$$

is called the Hessian curvature tensor for (D, g). The components of Q with respect to an affine coordinate system are given by

$$Q^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial x^k}.$$

Proposition 2.1.2 ([8]). *Let (M, D, g) be a Hessian manifold and $g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$, for some smooth function ϕ on M. Then we have*

(1) $Q_{ijkl} = \frac{1}{2} \frac{\partial^4 \phi}{\partial x^i \partial x^j \partial x^k \partial x^l} - \frac{1}{2} g^{rs} \frac{\partial^3 \phi}{\partial x^i \partial x^r \partial x^k} \frac{\partial^3 \phi}{\partial x^j \partial x^l \partial x^s}$,
(2) $Q_{ijkl} = Q_{kjil} = Q_{klij} = Q_{jilk}$.

Proposition 2.1.3 ([8]). *Let (M, D, g) be a Hessian manifold and R the Riemannian curvature tensor for g. Then*

$$R_{ijkl} = \frac{1}{2}(Q_{ijkl} - Q_{jikl}).$$

Definition 2.1.3 ([8]). Let (D, g) be a Hessian structure and ν the volume element of g. We define a closed 1-form α and a symmetric 2-form β by

$$D^\nu_X = \alpha(X)\nu,$$

$$\beta = D\alpha,$$

then α and β are called **the first Koszul form** and **the second Koszul form** for (D, g) respectively.

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