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Hamilton–Jacobi theorems for regular reducible Hamiltonian systems on a cotangent bundle

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ABSTRACT

In this paper, some of formulations of Hamilton–Jacobi equations for Hamiltonian system and regular reduced Hamiltonian systems are given. At first, an important lemma is proved, and it is a modification for the corresponding result of Abraham and Marsden (1978), such that we can prove two types of geometric Hamilton–Jacobi theorem for a Hamiltonian system on the cotangent bundle of a configuration manifold, by using the symplectic form and dynamical vector field. Then these results are generalized to the regular reducible Hamiltonian system with symmetry and momentum map, by using the reduced symplectic form and the reduced dynamical vector field. The Hamilton–Jacobi theorems are proved and two types of Hamilton–Jacobi equations, for the regular point reduced Hamiltonian system and the regular orbit reduced Hamiltonian system, are obtained. As an application of the theoretical results, the regular point reducible Hamiltonian system on a Lie group is considered, and two types of Lie–Poisson Hamilton–Jacobi equation for the regular point reduced system are given. In particular, the Type I and Type II of Lie–Poisson Hamilton–Jacobi equations for the regular point reduced rigid body and heavy top systems are shown, respectively.

1. Introduction

Symmetry is a general phenomenon in the natural world, but it is widely used in the study of mathematics and mechanics. The reduction theory for mechanical system with symmetry has its origin in the classical work of Euler, Lagrange, Hamilton, Jacobi, Routh, Liouville and Poincaré and its modern geometric formulation in the general context of symplectic manifolds and equivariant momentum maps is developed by Meyer, Marsden and Weinstein; see Abraham and Marsden [1] or Marsden and Weinstein [2] and Meyer [3]. The main goal of reduction theory in mechanics is to use conservation laws and the associated symmetries to reduce the number of dimensions of a mechanical system required to be described. So, such reduction theory is regarded as a useful tool for simplifying and studying concrete mechanical systems. Hamiltonian reduction theory is one of the most active subjects in the study of modern analytical mechanics and applied mathematics, in which a lot of deep and beautiful results have been obtained, see the studies given by Abraham and Marsden [1], Arnold [4], Marsden et al. [5–7,2], Ortega and Ratiu [8], Libermann and Marle [9], León and Rodrigues [10], etc. on regular point reduction and regular orbit reduction, singular point reduction and singular orbit reduction, optimal reduction and reduction by stages for Hamiltonian systems and so on; and there is still much to be done in this subject.

At the same time, we note also that the well-known Hamilton–Jacobi theory is an important part of classical mechanics. On the one hand, Hamilton–Jacobi equation provides a characterization of the generating functions of certain

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time-dependent canonical transformations, such that a given Hamiltonian system in such a form that its solutions are extremely easy to find by reduction to the equilibrium, see Abraham and Marsden [1], Arnold [4] and Marsden and Ratiu [7]. On the other hand, it is possible in many cases that Hamilton–Jacobi equation provides an immediate way to integrate the equation of motion of system, even when the problem of Hamiltonian system itself has not been or cannot be solved completely. In addition, the Hamilton–Jacobi equation is also fundamental in the study of the quantum–classical relationship in quantization, and it also plays an important role in the development of numerical integrators that preserve the symplectic structure and in the study of stochastic dynamical systems, see Woodhouse [11], Ge and Marsden [12], Marsden and West [13] and Lázaro-Camí and Ortega [14]. For these reasons Hamilton–Jacobi theory is described as a useful tool in the study of Hamiltonian system theory, and has been extensively developed in past many years. We note that some beautiful results have been obtained, see Cariñena et al. [15] and [16], Iglesias et al. [17], for more details.

Now, it is a natural problem how to study the Hamilton–Jacobi theory for a variety of reduced Hamiltonian systems by combining with reduction theory and Hamilton–Jacobi theory of Hamiltonian systems. This is a goal of our research. In this paper, some of formulations of Hamilton–Jacobi equations for Hamiltonian system and regular reduced Hamiltonian systems are given, and the main contributions are as follows: (1) We prove a key lemma, which is an important tool for proofs of the following theorems; (2) We prove two types of geometric Hamilton–Jacobi theorem for a Hamiltonian system on the cotangent bundle of a configuration manifold, by using the symplectic form and dynamical vector field; (3) We generalize the above results to the regular reducible Hamiltonian system with symmetry, and obtain two types of Hamilton–Jacobi equations for the regular point reduced Hamiltonian system and the regular orbit reduced Hamiltonian system, see Theorems 3.3, 3.4, 4.2 and 4.3, by using the reduced symplectic forms and the reduced dynamical vector fields; It is worthy of noting that the regular reduced symplectic spaces of the regular orbit reduced Hamiltonian system and the regular point reduced Hamiltonian system are different, and the symplectic forms on the reduced spaces are also different. Thus, the assumption conditions in Theorems 4.2 and 4.3 are different from the assumption conditions in Theorems 3.3 and 3.4, which depend on the precise analysis of the geometric structures of the regular orbit reduced space; (4) As an application, we give two types of Lie–Poisson Hamilton–Jacobi equation for the regular point reduced Hamiltonian system on a Lie group, and show the Type I and Type II of Lie–Poisson Hamilton–Jacobi equations for the regular point reduced rigid body and heavy top systems, respectively. In general, we know that it is not easy to find the solutions of Hamilton's equation. But, if we can get a solution of Hamilton–Jacobi equation for a Hamiltonian system, by using the relationship between Hamilton's equation and Hamilton–Jacobi equation, it is easy to give a special solution of Hamilton's equation. Thus, it is very important to give explicitly the various formulations of Hamilton–Jacobi equations for Hamiltonian system and the reduced Hamiltonian systems.

A brief outline of this paper is as follows. In the second section, we first prove a key lemma, which is obtained by a careful modification for the corresponding results of Abraham and Marsden in [1]. Then we prove two types of geometric version of Hamilton–Jacobi theorem of a Hamiltonian system on the cotangent bundle of a configuration manifold, by using the symplectic form and the dynamical vector field. In the third section and the fourth section, we discuss the regular reducible Hamiltonian systems with symmetry and momentum map, by combining with the Hamilton–Jacobi theory and the regular symplectic reduction theory. The two types of Hamilton–Jacobi equations for the regular point and the regular orbit reduced Hamiltonian systems are obtained, respectively, by using the reduced symplectic forms and the reduced dynamical vector fields. As the applications of the theoretical results, in the fifth section, the regular point reducible Hamiltonian system on a Lie group is considered, and two types of Lie–Poisson Hamilton–Jacobi equation for the regular point reduced system are given. In particular, the Type I and Type II of Lie–Poisson Hamilton–Jacobi equations for the regular point reduced rigid body and heavy top systems are shown, respectively. These research works develop the reduction and Hamilton–Jacobi theory of a Hamiltonian system with symmetry and make us have much deeper understanding and recognition for the structures of Hamiltonian systems.

2. Geometric Hamilton–Jacobi theorem of Hamiltonian system

In this section, we first review briefly some basic facts about Hamilton–Jacobi theory, and state our idea to study the problem in this paper. Then we prove a key lemma, which is an important tool for the proofs of geometric Hamilton–Jacobi theorems of Hamiltonian system and the regular reducible Hamiltonian system with symmetry. Finally, we prove two types of geometric version of Hamilton–Jacobi theorem of a Hamiltonian system on the cotangent bundle of a configuration manifold, by using the symplectic form and dynamical vector field. It is worthy of noting that we describe the Hamilton–Jacobi equation by Hamiltonian vector field of the system, it is easy to be generalized to the cases of the regular reduced Hamiltonian systems. We shall follow the notations and conventions introduced in Abraham and Marsden [1], Marsden and Ratiu [7], Ortega and Ratiu [8], and Marsden et al. [18]. In this paper, we assume that all manifolds are real, smooth and finite dimensional and all actions are smooth left actions.

It is well-known that Hamilton–Jacobi theory from the variational point of view is originally developed by Jacobi in 1866, which states that the integral of Lagrangian of a system along the solution of its Euler–Lagrange equation satisfies the Hamilton–Jacobi equation. The classical description of this problem from the geometrical point of view is given by Abraham and Marsden in [1] as follows: Let Q be a smooth manifold and TQ the tangent bundle, T^*Q the cotangent bundle with the canonical symplectic form ω and the projection $\pi_Q : T^*Q \to Q$ induces the map $T\pi_Q : TT^*Q \to TQ$.